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# On the Motive of the Intersection Cohomology of a Singular Three Dimensional Variety

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# Contents

# Introduction

This thesis is devoted to the study of two different topics. The first part deals with the motive of the intersection cohomology of a singular three dimensional complex projective variety and it is developed in Chapter 2. Chapters 3 and 4 concern the second subject which consists in the study of the compactification of the configuration space of Fulton and MacPherson.

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The first topic concerns with the intersection cohomology of a singular variety.

The intersection cohomology $IH^*(Y)$ is introduced by Goresky and MacPherson in [15] for the study of a (singular) algebraic variety $Y$ with the purpose to show that special properties of the singular cohomology of nonsingular varieties can be extended to the singular case.

Roughly speaking, the intersection homology groups are defined as certain cycles modulo certain homologies, the cycles and homologies being restricted as to how they meet the singular set of $Y$ (see [15]). Developing some ideas of Deligne and Verdier, Goresky and MacPherson in [16] give an algebraic approach to the construction of the intersection cohomology $IH^*(Y)$ as the hypercohomology of a complex of sheaves on the variety.

Let us recall several properties of the rational intersection cohomology groups for a projective complex variety $Y$ of dimension $n$ (see Cheeger, Goresky and MacPherson [4] and Goresky and MacPherson [15], [16], and [17]):

1. the groups $IH_i(Y)$ are topological invariants but not homotopy invariants, for $i = 0, \dots, 2n$;
2. (Poincaré duality) there is an intersection product $IH^i(Y) \times IH^j(Y) \to H_{2n-i-j}(Y)$ which is non-degenerate for $j = 2n - i$ and therefore it induces isomorphisms $IH^i(Y) \cong IH^{2n-i}(Y)$;
3. there exists a factorization of the Poincaré map $\cap[Y] : H^i(Y) \to H_{2n-i}(Y)$
$$H^i(Y) \to IH^i(Y) \to H_{2n-i}(Y)$$
and if $Y$ is nonsingular the maps are isomorphisms;
4. (Hard Lefschetz) if $A$ is an ample line bundle on $Y$ then the cup product map $A^i : IH^{n-i}(Y) \to IH^{n+i}(Y)$ is an isomorphism for all $i \geq 0$;

5. (the Lefschetz hyperplane theorem) if $Y$ is embedded in a projective space and $Y'$ is a generic hyperplane section, then $IH^i(Y') \to IH^i(Y)$ is an isomorphism for $i < n-1$ and is an injection for $i = n-1$.

A further property shared by the singular cohomology and the intersection cohomology is proved by Saito in [33]. He shows that the intersection cohomology $IH^*(Y)$ of a projective variety carries a natural pure Hodge structure. Recently, de Cataldo and Migliorini in [8] establish that the pure Hodge structure of $IH^*(Y)$ is induced by the Hodge structure on the cohomology of any resolution of the singularities of $Y$: given a resolution $f : X \to Y$, the space $IH^i(Y)$ is included in a quotient of $H^i(X)$ and it inherits a weight-$i$ pure Hodge structure. Therefore

6. the intersection cohomology groups $IH^*(Y)$ have a pure Hodge structure, i.e. there exists a direct sum decomposition

$$IH^i(Y, \mathbb{C}) \cong \bigoplus_{p+q=i} IH^{(p,q)}(Y)$$

with $IH^{(p,q)}(Y) \cong \overline{IH^{(q,p)}(Y)}$, and this structure is induced by the Hodge structure on the singular cohomology $H^*(X)$ of any resolution $X$ of $Y$.

In this thesis we deal with a problem related to this last property. It is conjectured (see Corti and Hanamura [5]) that pure Hodge structures arise as the cohomology groups of a pure motive.

We recall that a pure homological (resp. Chow) motive is a triple $(X, P, r)$ where $X$ is a nonsingular projective variety, $P$ is a homological (resp. Chow) projector in $X \times X$ and $r$ is an integer. We will consider motives for which $r = 0$, for this reason we will drop it from the notation. We can define, for instance, the cohomology with coefficients in $\mathbb{Q}$ of the motive $(X, P)$ by $H^i((X, P), \mathbb{Q}) := P_* H^i(X, \mathbb{Q})$, where $P_*$ is the homomorphism induced by $P$. Thus, such motives are called pure because their realization through an adequate cohomology theory has a pure Hodge structure.

In this context, Corti and Hanamura [5] refer to the conjectural pure motive associated with the intersection cohomology $IH^*(Y)$ as the "intersection motive" of the variety $Y$.

From this point of view, we investigate the case of a three dimensional complex projective variety $Y$ with an isolated singular point $y$ and assume that $f : X \to Y$ is a resolution of the singularity of $Y$ with $D = f^{-1}(y)$ a smooth exceptional divisor. The case of several isolated singular points is entirely similar.

As a result of our investigation, we find a homological motive which realizes the intersection cohomology $IH^*(Y, \mathbb{Q})$ and we show that this motive is a direct summand of the motive of the desingularization $(X, cl(\Delta))$, where $cl(\Delta) \in H_6(X \times X)$ is the class of the diagonal, as conjectured by Corti and Hanamura [5].

Going into details, in order to define the required motive, we analyse each group $IH^i(Y)$. To do this, we make use of a result of de Cataldo and Migliorini [8] which realizes in cohomology the topological decomposition theorem applied to $f$ (see Corti and Hanamura [5])

$$Rf_*\mathbb{Q}_X = V \oplus \mathcal{IC}_Y$$

where $V$ is a sheaf supported on the singular point and $\mathcal{IC}_Y$ is the intersection complex of $Y$ whose hypercohomology groups are the intersection cohomology groups of $Y$. The result in [8] gives an explicit decomposition of the rational cohomology groups of the desingularization $X$ in terms of the cohomology groups of the exceptional divisor $D$ and of the intersection cohomology groups of the singular variety $Y$ (cf. Theorem 2.1.4 of the thesis).

Specifically, as regards the groups $IH^i(Y)$ with $i = 0, 1, 5, 6$, the decompositions give isomorphisms with the corresponding groups $H^i(X)$.

Concerning $IH^i(Y)$, for $i = 0, 6$, we are led to consider the well-known trivial motives (see for instance Kleiman [23]), which are defined as $(X, e \times X)$ and $(X, X \times e)$, being $e$ a point of $X$. We deduce that the groups $IH^0(Y)$ and $IH^6(Y)$ are the cohomology groups of the trivial motives as follows (see Proposition 2.2.3)

$$IH^0(Y) = H^*(X, (e \times X)) \qquad IH^6(Y) = H^*(X, (X \times e))$$

Analogously, the isomorphisms $IH^i(Y) \cong H^i(X)$, for $i = 1, 5$, allow us to obtain these intersection cohomology groups as the cohomology groups of the so called Picard and Albanese motives (these are defined by Kleiman [22] in the homological case and are studied by Murre [30] in the algebraic one):

$$IH^1(Y) = H^*(X, \frac{1}{n}\,\tilde{E}) \qquad IH^5(Y) = H^*(X, \frac{1}{n}\,{}^t\tilde{E})$$

(see Proposition 2.2.6).

The remaining three intersection cohomology groups are direct summands of the cohomology of $X$, hence we have to devise a new strategy to search for motives which realize such groups.

As regards the group $IH^2(Y)$, it is a direct summand of $H^2(X)$, indeed $H^2(X) = H_4(D) \oplus IH^2(Y)$. In this case, by fixing an ample line bundle on $X$, we are able to define an algebraic cycle $\gamma_2$ in $X \times X$ which is supported on $D \times D$ and which turns out to be a Chow (and hence a homological) projector. We show that it induces the projection from the cohomology of $X$ to the group $H_4(D)$ (see Theorem 2.3.1) and, from this fact, the group $IH^2(Y)$ is the cohomology of a homological motive as follows (see Proposition 2.3.2)

$$IH^2(Y) = H^*(X, \pi_{2*} - cl(\gamma_2)_*)$$

where $\pi_2$ is the $(4, 2)$-Künneth component of the homological class of the diagonal of $X \times X$ and $cl : A^i(X \times X) \to H^{2i}(X \times X)$ is the cycle class map. By means of an analogous procedure, we exhibit an algebraic cycle $\gamma_4$ which projects $H^*(X)$ onto $H^4(D)$ (see Theorem 2.3.2) and such that

$$IH^4(Y) = H^*(X, \pi_{4*} - cl(\gamma_4)_*)$$

where $\pi_4 \in H^2(X) \otimes H^4(X)$ is the $(2,4)$-Künneth component of $cl(\Delta)$ (see Proposition 2.3.3).

Finally, the group $IH^3(Y)$ is a direct summand of $H^3(X)$ which splits as a direct sum of $IH^3(Y)$ and $H^3(D)$. In order to project the cohomology of $X$ onto $H^3(D)$, we realize that we have to investigate the Albanese motive on the exceptional divisor $D$ for having a good candidate. We then construct an algebraic cycle $\gamma_3$ with the required property (see Theorem 2.4.2). Nevertheless, in order to have a homological projector, we notice that we have to assume the dual of the normal bundle to $D$ in $X$ is ample, or at least nef and big. Under this hypothesis, we show that the intersection cohomology group $IH^3(Y)$ is the cohomology of a homological motive (see Proposition 2.4.3):

$$IH^3(Y) = H^*(X, \pi_{3*} - cl(\gamma_3)_*)$$

where $\pi_3 \in H^3(X) \otimes H^3(X)$ is a Künneth component of $cl(\Delta)$.

Collecting these results we obtain that the intersection cohomology $IH^*(Y)$ is the cohomology of the homological motive $(X, \gamma)$, for an appropriate $\gamma \in H^6(X \times X)$, provided the dual of the normal bundle $N_D X$ is at least nef and big (cf. Theorem 2.4.3). In addition $\gamma$ is a summand for a decomposition of $cl(\Delta)$ and therefore the intersection homological motive $(X, \gamma)$ is a direct summand of the motive $(X, cl(\Delta))$ of the desingularization, as predicted by the conjectures.

We provide an application of the results to the case of elliptic modular threefolds, namely we determine a Chow motive whose cohomology is the intersection cohomology of a singular threefold (see Theorem 2.5.1).

We conclude Chapter 2 with a generalization of the previous construction to the case of a variety $Y$ which is singular in an isolated point and along a nonsingular curve $C$ on which the resolution $f$ is locally topologically trivial with 1-dimensional fibers. By applying the motivic decomposition for semismall maps (see de Cataldo and Migliorini [6]) we derive an analogous statement for this case (see Theorem 2.6.2).

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The second topic we treat in the thesis is the study of the compactification of the configuration space of Fulton and MacPherson [13] which is usually indicated by $X[n]$. Some of the reasons making this space worth of special interest are the following: the relation between the case $X = \mathbb{P}^1_{\mathbb{C}}$ and the Knudsen compactification of the moduli space $\mathcal{M}_{0,n}$ of compact complex algebraic curves of genus 0 with $n$ marked points (see [25]); the resolution of the singularities of $X^n/S_n$ for varieties of dimension greater than two; the operad structure; the analogous construction in the real case which M. Kontsevich studied for any compact manifold $X$ (see [26]).

Given a non-singular algebraic variety $X$ over an algebraically closed field, the configuration space of $n$ distinct labeled points on $X$ is by definition $F(X, n) = X^n \setminus \bigcup \Delta_{\{a,b\}}$. It is the complement in the cartesian product of the large diagonals $\Delta_{\{a,b\}}$ where points with labels $a$ and $b$ are equal. There is a natural embedding

$$F(X, n) \subset X^n \times \prod_{|S| \geq 2} Bl_\Delta(X^S)$$

where $S$ is a subset of $\{1, \dots, n\}$ with at least two elements and $Bl_\Delta(X^S)$ is the blow-up of the corresponding cartesian product $X^S$ along its small diagonal.

The compactification $X[n]$ is defined to be the closure of the configuration space inside the product.

The following are some of the basic properties characterizing such a variety:

1. $X[n]$ is non-singular;
2. the canonic map from $X[n]$ to $X^n$ is proper;
3. the complement $X[n] \setminus F(X, n)$ is a normal crossing divisor;
4. the symmetric group $S_n$ acts on $X[n]$ and the isotropy group of each point is solvable.

All of these results and much more is contained in [13]. In the same paper, Fulton and MacPherson give an explicit construction of $X[n]$ starting from $X^n$ through a sequence of blow-ups defined by induction on $n$. Roughly speaking, to obtain $X[n]$ one blow-ups $X^n$ along all the diagonals according to a precise order among the indexes $\{1, \dots, n\}$. For more details we refer to Chapter 1.

Our first task consists in the determination of the Chow groups of $X[n]$

$$A_k(X[n]) = Z_k(X[n])/Rat_k(X[n]), \quad k \in \mathbb{Z}$$

($k$-cycles modulo rational equivalence).

These Chow groups turns out to be direct sums of Chow groups of the exceptional divisors and of the starting variety $X^n$. Since the exceptional divisors are projective bundles we can go further. By exploiting the relationship between the Chow groups of a smooth blow-up and the Chow groups of the centers and of the blown-up variety, we deduce a formula expressing the Chow groups of $X[n]$ in function of those of appropriate varieties $X[p] \times X^q$ of the same kind but with smaller invariants. From this we infer inductively that the Chow groups of $X[n]$ are direct sums of Chow groups of the product $X^r$, $r \leq n$ (see Theorem 3.2.2 and Corollaries 3.2.2 and 3.2.3).

These results are stated in Chapter 3, where several other properties are established: an explicit development of the Chow groups for $n = 2, 3, 4$; a numeric reformulation of Theorem 3.2.2; a description of the exceptional divisors in the case of non-singular curves and the determination of the invariant Neron-Severi group in the projective case; a functorial description of the Zariski tangent space to $X[n]$.

In [13] Fulton and MacPherson express the Chow ring of the compactification $X[n]$ as a polynomial ring

$$\bigoplus_k A_k(X[n]) = A^*(X[n]) = A^*(X^n)[D_S]/I$$

where the variables $D_S$ are the exceptional divisors and $I$ is an ideal of relations among the divisors $D_S$ depending on $S \subseteq \{1, \ldots, n\}$. However, the single Chow groups remain quite mysterious.

By specializing to the case $X = \mathbb{C}^m$, in Chapter 4, we are able to find an explicit basis of the Chow groups in the case of $n = 2, 3, 4$. Moreover, we compute their dimension and with a careful analysis we discover a symmetry between such groups with respect to the codimension $[\frac{(n-1)m-1}{2}]$ (see Theorems 4.1.1, 4.2.1 and 4.3.1). Indeed

$$A^p(\mathbb{C}^m[n]) \cong A^{(n-1)m-1-p}(\mathbb{C}^m[n])$$

for all $p$, $0 \leq p \leq [\frac{(n-1)m-1}{2}]$.

For $n \leq 3$ we are also able to give explicit isomorphisms which realize the symmetry. Such isomorphisms are obtained by intersecting with cycles expressed by the exceptional divisors (see Theorems 4.1.1, 4.2.1). It is reasonable to conjecture that these properties hold for all $n$, but they seem to be completely out of reach by direct computations. Maybe more sophisticated combinatorial tools could avoid the maze of calculations and shed a new light on this intriguing subject.

# Chapter 1

# Preliminaries

This chapter provides the basic notions that we will need in the sequel. In particular, section 1.1 contains tools useful for both the topics studied in this thesis, sections 1.2 and 1.3 cover some topics of Chapter 2, whereas section 1.4 deals with the subject of Chapters 3 and 4.

## 1.1 Intersection theory

In this section we define the Chow group and expose some of its properties. The main source for our notations and results is the beautiful book of Fulton [12].

By *scheme* we mean an algebraic scheme over a field. By *variety* we mean a reduced and irreducible algebraic scheme and a *subvariety* of a scheme is a closed subscheme which is a variety.

Let $X$ be an algebraic scheme over a field. A $k$-cycle on $X$ is a finite formal sum

$$\sum n_i[V_i]$$

where the $V_i$ are $k$-dimensional subvarieties of $X$, $n_i$ are integers. The group of $k$-cycles on $X$ is the free abelian group on the $k$-dimensional subvarieties of $X$ and it is denoted $Z_kX$.

For any $(k+1)$-dimensional subvariety $W$ of $X$ and for any non-zero rational function on $W$, $r \in R(W)^*$, define a $k$-cycle on $X$ by

$$[div(r)] = \sum ord_V(r)[V]$$

the sum running over all codimension 1 subvarieties $V$ of $W$ and $ord_V$ being the order function on $R(W)^*$ defined by the local ring $\mathcal{O}_{V,W}$.

A $k$-cycle $\alpha$ is *rationally equivalent to zero* if there are a finite number of $(k+1)$-dimensional subvarieties $W_i$ of $X$ and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [div(r_i)]$$

and we write $\alpha \sim 0$. The cycles which are rationally equivalent to zero form a subgroup of $Z_kX$ because $[div(r^{-1})] = -[div(r)]$, and we indicate this group by $Rat_kX$.

**Definition 1.1.1.** *The group of $k$-cycles on $X$ modulo rational equivalence is the quotient group*

$$A_kX := Z_kX/Rat_kX$$

*It is also called the Chow group of order $k$ on $X$ and also denoted by $CH_kX$. The direct sum of $Z_kX$ (resp. $A_kX$) for $k = 0, \dots, \dim X$ is denoted by $Z_*X$ (resp. $A_*X$). We will use the notation $A^pX := A_{\dim X - p}X$ for cycles of codimension $p$ and $A^*X = \oplus_p A^pX$.*

If $f : X \to Y$ is a flat morphism of relative dimension $n$ then there are induced morphisms, the *flat pull-backs*,

$$f^* : A_kY \to A_{k+n}X$$

that send $[W] \in Z_kY$ to $[f^{-1}(W)]$.

If $f : X \to Y$ is a proper morphism then there are *push-forwards*,

$$f_* : A_kX \to A_kY$$

which send $V \in Z_kX$ to $\deg(V/f(V))[f(V)]$, where

$$\deg(V/f(V)) = \begin{cases} [R(V) : R(f(V))] & \text{if } \dim(V) = \dim(f(V)) \\ 0 & \text{if } \dim(V) < \dim(f(V)) \end{cases}$$

In an alternate and more classical definition, $k$-cycles are rational equivalent to zero if generated by $[V(0)] - [V(\infty)]$ for $(k+1)$-dimensional subvarieties $V$ of $X \times \mathbb{P}^1$ such that the projection to the second factor induces a dominant morphism $f$ from $V$ to $\mathbb{P}^1$, where $[V(P)] := p_*[f^*(P)]$ for $P \in \mathbb{P}^1$ and $p : X \to X \times \mathbb{P}^1$ is the projection to $X$.

If $X$ is a non-singular variety of dimension $n$, the product

$$A^pX \otimes A^qX \to A^{p+q}X : x \otimes y \mapsto x \cdot y$$

makes $A^*X$ into a commutative, associative ring with unit $[X] \in A_nX$ and it is called *the intersection ring* of $X$ (see Proposition 8.3 of [12]). The product $x \cdot y$ is the intersection product defined by

$$x \cdot y = \delta^*(x \times y)$$

where $\delta : X \to X \times X$ is the diagonal embedding and $\delta^*$ is the Gysin homomorphism (see §6.2 of [12]).

In a more general context, given a regular embedding $i : X \to Y$ of codimension $d$ and a morphism $f : V \to Y$ from a pure $k$-dimensional scheme, it is defined an intersection product $X \cdot V$ constructed in $A_{k-d}(f^{-1}(X))$. We briefly recall its construction. The normal cone $C_{f^{-1}(X)}V$ is a closed subcone of the pull-back of the normal bundle $N_XY$

to $f^{-1}(X)$, of pure dimension $k$. We define $X \cdot V$ to be the result of intersecting the cycle $[C_{f^{-1}(X)}V]$ by the zero section of this bundle, i.e. if $s : f^{-1}(X) \to f^*N_XY$ is the zero section, then $X \cdot V = s^*[C_{f^{-1}(X)}V]$ is the cycle class in $A_{k-d}(f^{-1}(X))$ corresponding to $[C_{f^{-1}(X)}V]$ through the isomorphism given by the flat pull-back of the projection $\pi : f^*N_XY \to f^{-1}(X)$ (see the next statement).

The following result connects the Chow groups of a projective bundle with those of the base.

Let $E$ be a vector bundle of rank $d$ on a scheme $X$ with projection $\pi : E \to X$. Let $\mathbb{P}(E)$ be the associated projective bundle and $p$ the projection from $\mathbb{P}(E)$ to $X$ and $\mathcal{O}(1)$ the canonical line bundle on $\mathbb{P}(E)$. Then

**Theorem 1.1.1.** *1. The flat pull-back*

$$\pi^* : A_{k-d}X \to A_kE$$

*is an isomorphism for all $k$.*

*2. Each element $\beta \in A_k(\mathbb{P}(E))$ is uniquely expressible in the form*

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i$$

*for $\alpha_i \in A_{k-d+1+i}(X)$ where $c_1(\mathcal{O}(1))$ is the first Chern class of the line bundle and $\cap$ is the cap product (see for istance §2.5 of [12]). Thus there are canonical isomorphisms*

$$A_k(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{d-1} A_{k-i}(X). \tag{1.1}$$

For a proof of this theorem see Theorem 3.3 of [12].

Now let $X$ be a regularly imbedded subscheme of a scheme $Y$ of codimension $d$ with normal bundle $N = N_XY$. Let $\mathbb{P}(N)$ be the exceptional divisor of the blowup of $Y$ along $X$. There is a fiber square

$$\begin{array}{ccc} \mathbb{P}(N) & \xrightarrow{j} & Bl_XY \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Then there are split exact sequences

$$0 \to A_kX \xrightarrow{\alpha} A_k\mathbb{P}(N) \oplus A_kY \xrightarrow{\beta} A_kBl_XY \to 0$$

with $\alpha(x) = (c_{d-1}(E) \cap g^*x, -i_*x)$ ($E = g^*N/N_{\mathbb{P}(N)}(Bl_XY)$ is the universal quotient bundle on $\mathbb{P}(N)$) and $\beta(\tilde{x}, y) = j_*\tilde{x} + f^*y$ (see Proposition 6.7 of [12]).

From these exact sequences and formula (1.1), by eliminating a copy of $A_kX$ in $A_k\mathbb{P}(N)$ we get the following isomorphisms

$$A_k(Bl_XY) \cong \bigoplus_{i=1}^{d-1} A_{k-i}(X) \oplus A_k(Y) \tag{1.2}$$

## 1.2 Chow motives

The topics we briefly expose in this section consist in several basic notions regarding mainly Chow motives. They are included as part of the preparatory tools to the development of the arguments studied in Chapter 2.

The main source we refer to are the works of Corti and Hanamura [5], Fulton [12], Kleiman [22], [23], and [24], and Murre [30].

Consider a non-singular projective $n$-dimensional complex variety $X$.

To fix the notations, the cohomology groups of $X$ are the singular cohomology groups with rational coefficients $H^i(X) = H^i(X, \mathbb{Q})$. Such groups are zero for $i \notin [0, 2n]$ and satisfy Poincaré duality thus we have the identifications $H^i(X) = H_{2n-i}(X)$, $H^i(X) \cong H^{2n-i}(X)^*$.

If $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the two projections then the map $a \otimes b \mapsto p_X^* a \cdot p_Y^* b$ is an isomorphism

$$H^*(X) \otimes H^*(Y) \xrightarrow{\cong} H^*(X \times Y) \qquad \text{(the Künneth isomorphism)}$$

To each subvariety of $X$ of dimension $k$ we can associate a class in $H_{2k}(X)$ and since rationally equivalent to zero cycles are homologically equivalent to zero there is a homomorphism from $A_k(X)$ to $H_{2k}(X)$ which after identifications reads as follows

$$cl : A^i(X) \to H^{2i}(X) \qquad \text{(the cycle class map)}$$

This map is functorial, i.e. if $f : X \to Y$ is a homomorphism then

$$f^* cl_Y = cl_X f^* \qquad cl_Y f_* = f_* cl_X$$

Recall that according to an adequate equivalence relation on the group of algebraic cycles on $X$ of codimension $i$, $Z^i(X)$, we obtain quotient groups such as:

i) $A^i(X)$, the Chow groups if we consider rational equivalence;

ii) $H^i_{alg}(X) = Im(A^i(X) \xrightarrow{cl} H^{2i}(X))$, the algebraic cycles modulo homological equivalence which are called algebraic cohomology cycles;

iii) $C^i(X)$ (or $N^i(X)$), the cycles modulo numerical equivalence (a $i$-cocycle $\alpha$ is numerically equivalent to zero if $\deg(\beta \cdot \alpha) = 0$, for all $\beta \in Z^{n-i}(X)$).

According to the various equivalence relations we can define respectively

i) Chow motives;

ii) homological motives;

iii) Grothendieck motives.

In order to define a motive we need the notion of correspondence:

**Definition 1.2.1.** *A correspondence from a non-singular projective variety $X$ of dimension $n$ to a non-singular projective variety $Y$ of dimension $m$ is an equivalence class in $X \times Y$, for example in $A^*(X \times Y)$ (it is also denoted by $\alpha : X \vdash Y$).*

For relative correspondences, supported on fiber products, see [5] or [6].

If $X, Y$ and $Z$ are non-singular projective varieties and if $\alpha \in A^*(X \times Y)$ is a correspondence from $X$ to $Y$ and $\beta \in A^*(Y \times Z)$ is a correspondence from $Y$ to $Z$ it is defined the *composite* (or *product*) correspondence $\beta \circ \alpha \in A^*(X \times Z)$

$$\beta \circ \alpha = p_{XZ*}(p_{XY}^* \alpha \cdot p_{YZ}^* \beta)$$

where $p_{XZ}, p_{XY}, p_{YZ}$ are the projections from $X \times Y \times Z$ to $X \times Z$, $X \times Y$ and $Y \times Z$ respectively and $p_{XY}^* \alpha \cdot p_{YZ}^* \beta$ is the intersection product on the non-singular variety $X \times Y \times Z$ (see §1.1 or [12]).

A morphism $f : X \to Y$ determines a correspondence from $X$ to $Y$, its graph $\Gamma_f$; the *transpose* of a correspondence $\alpha$ is defined to be $\tau_*(\alpha)$, where $\tau : X \times Y \to Y \times X$ reverses the factors, i.e. $\tau(x, y) = (y, x)$, the transpose is indicated with $\alpha'$ or ${}^t\alpha$.

A correspondence $\alpha \in A^*(X \times Y)$ determines a homomorphism

$$\alpha_* : A^*(X) \to A^*(Y) : a \mapsto p_{Y*}^{XY}(\alpha \cdot p_X^{XY*}(a))$$

and a homomorphism

$$\alpha^* : A^*(Y) \to A^*(X) : b \mapsto p_{X*}^{XY}(\alpha \cdot p_Y^{XY*}(b))$$

If $\alpha \in A^*(X \times Y)$, $\beta \in A^*(Y \times Z)$ and $\gamma \in A^*(Z \times W)$ are correspondences, $\Delta_X \in A^n(X \times X)$ is the class of the diagonal of $X \times X$ and $f : X \to Y$ is a homomorphism then the following properties are satisfied:

$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha \tag{1.3}$$
$$\alpha \circ \Delta_X = \alpha \qquad \Delta_Y \circ \alpha = \alpha \tag{1.4}$$
$$(\beta \circ \alpha)' = \alpha' \circ \beta' \qquad (\alpha')' = \alpha \tag{1.5}$$
$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_* \qquad (\beta \circ \alpha)^* = \alpha^* \circ \beta^* \tag{1.6}$$
$$(\alpha')_* = \alpha^* \qquad (\Gamma_f)_* = f_* \qquad (\Gamma_f)^* = f^* \tag{1.7}$$

In particular the class of the diagonal $\Delta_X \in A^n(X \times X)$ induces the identity map.

If $\alpha \in H^*(X \times Y)$ is a homological correspondence we analogously define the homomorphisms $\alpha_* : H^*(X) \to H^*(Y)$ and $\alpha^* : H^*(Y) \to H^*(X)$. Because of the following isomorphisms

$$\begin{aligned} H^*(X \times Y) \cong & H^*(X) \otimes H^*(Y) && \text{(Künneth formula)} \\ \cong & H_*(X) \otimes H^*(Y) && \text{(Poincaré duality)} \\ \cong & Hom(H^*(X), H^*(Y)) \end{aligned}$$

a homological correspondence is the same as a homomorphism between cohomology groups.

**Definition 1.2.2.** *A correspondence $\alpha \in A^*(X \times Y)$ is homogeneous of degree $p$ if $\alpha \in A^{m+p}(X \times Y)$, where $m = \dim Y$.*

If $\alpha : X \vdash Y$ has degree $p$ and $\beta : Y \vdash Z$ has degree $q$ then $\beta \circ \alpha$ has degree $p + q$, in particular correspondences of degree 0 are closed under composition.
If $\alpha : X \vdash Y$ has degree $p$ then $\alpha_* : A_k(X) \to A_{k-p}(Y)$ and $\alpha^* : A^k(Y) \to A^{k+p}(X)$.

In the homological case:

1. if $\alpha \in H^p(X \times Y)$ then $\alpha_*$ is homogeneous of degree $p - 2n$;
2. if $\alpha$ is in the image of $H^{2n-i}(X) \otimes H^j(Y)$ then $\alpha_*$ vanishes on $H^p(X)$ when $p \neq i$ and maps $H^i(X)$ into $H^j(Y)$;
3. if $\Delta = \sum \pi_i$ is the Künneth decomposition of the diagonal of $X \times X$ and $\pi_i$ is in the image of $H^{2n-i}(X) \otimes H^i(X)$, then $\pi_{i*}$ is the projection of $H^*(X)$ onto $H^i(X)$.

**Definition 1.2.3.** *A correspondence $P \in A^{\dim X}(X \times X)$ is a projector if*

$$P^2 = P \circ P = P$$

We are now ready to define a motive:

**Definition 1.2.4.** *A Chow (resp. homological) motive is a triple*

$$(X, P, r)$$

*where $X$ is a non-singular projective variety, $P \in A^{\dim X}(X \times X)$ (resp. in $H^{2\dim X}(X \times X)$) is a projector and $r$ is an integer.*

Homomorphisms between Chow motives $(X, P, r)$ and $(Y, Q, s)$ are

$$Hom((X, P, r), (Y, Q, s)) = Q \circ (\oplus_a A^{\dim X_a + s - r}(X_a \times Y)) \circ P$$

where $X = \amalg_a X_a$ is the decomposition of $X$ into its connected components.

**Definition 1.2.5.** *There are defined the direct sum of two motives with the same integer*

$$(X, P, r) \oplus (Y, Q, r) = (X \cup Y, P \oplus Q, r)$$

*and the tensor product of two motives*

$$(X, P, r) \otimes (Y, Q, s) = (X \times Y, P \otimes Q, r + s)$$

If $r = 0$ we write $(X, P)$, if $P = \Delta$ we have a pure Chow motive.

**Definition 1.2.6.** *An effective motive is a couple $M = (X, P)$ where $X$ is a non-singular projective variety and $P \in A^{\dim X}(X \times X)$ such that $P^2 = P$ and homomorphisms are defined as*

$$Hom((X, P), (Y, Q)) = \{f \in A^{\dim X}(X \times Y); f \circ P = Q \circ f\}/\{f \circ P = Q \circ f = 0\}$$

In the category of effective motives every projector $P$ of $M$ has a kernel and gives a direct sum decomposition $M = ker(P) \oplus ker(id_M - P)$.

There is a controvariant functor $h$ which associate with $X$ the couple

$$h(X) = (X, id)$$

and to each homomorphism $\phi : X \to Y$ associates

$$h(\phi) = {}^t\Gamma_\phi \in A^{\dim Y}(Y \times X)$$

the transpose of the graph of $\phi$.

If $M = (X, P)$ is a motive then one defines the Chow groups of $M$ by

$$A^i(M, \mathbb{Q}) := P_* A^i(X, \mathbb{Q}) \subset A^i(X, \mathbb{Q})$$

and the cohomology groups

$$H^i(M, \mathbb{Q}) := P_* H^i(X, \mathbb{Q}) \subset H^i(X, \mathbb{Q})$$

If $\pi_i \in H^{2n-i}(X) \otimes H^i(X)$ is the $(2n-i, i)$-Künneth component of the (co)homology class of the diagonal $\Delta_X$ in $H^{2n}(X \times X)$, where $n = \dim X$, then $\pi_i$ is a homological projector and the decomposition $\Delta_X = \sum_i \pi_i$ induces a homology motivic decomposition

$$h(X) = \bigoplus_i h^i(X)$$

where $h^i(X) = (X, \pi_i)$.

## 1.3 The Albanese and the Picard varieties

The results we expose in this section are treated in various texts such as [28] of Lang, [29] of Lange-Birkenhake and partially in Griffiths-Harris [19]. They are quoted as part of the background material which we need in the development of Chapter 2.

In view of the discussion at Chapter 2 we reduce the study to the case of a non-singular projective complex variety $X$ of dimension $n$.

An *abelian variety* is a complex torus $V/\Lambda$ admitting a polarization $H = c_1(L)$, i.e. $L$ is an ample line bundle on $V/\Lambda$. The Riemann Relations are necessary and sufficient conditions for a complex torus to be an abelian variety; another characterization was given by Lefschetz who showed that a complex torus is an abelian variety if and only if it admits the structure of an algebraic variety (see the introduction to Chapter 4 of [29]).

In analogy to the case of curves, it is possible to associate to $X$ an abelian variety whose study gives information on the geometry of $X$. Such an abelian variety is called the Albanese variety (or Jacobian variety) of $X$.

**• The Albanese variety**

According to Lang (see $II$, §3 of [28]) *an Albanese variety* of $X$ is a couple $(A, f)$ where $A$ is an abelian variety and $f : X \to A$ is a rational map such that the two following conditions are satisfied:

i) $A$ is the smallest abelian subvariety of $A$ containing $f(A)$;

ii) *the universal property*: if $g : X \to B$ is a rational map into an abelian variety then there exist a unique homomorphism $h : A \to B$ and a constant $c \in B$ such that $g = hf + c$.

The map $f$ is called a canonical map. It can be proven that an Albanese variety $(A, f)$ exists for each variety $X$ and that $A$ is uniquely determined up to birational isomorphisms and $f$ up to a translation (see Theorem 11 at $II$, §3 of [28]).

Under our hypothesis the Albanese variety of $X$ can be defined alternatively as follows (see [19] and [29]). Each element $\gamma \in H_1(X, \mathbb{Z})$ determines a homomorphism on the vector space of holomorphic 1-forms on $X$

$$\begin{aligned} \gamma : H^0(X, \Omega^1_X) &\to \mathbb{C} \\ \omega \qquad\qquad &\mapsto \int_\gamma \omega \end{aligned}$$

The homology group $H_1(X, \mathbb{Z})$ is a lattice in $H^0(X, \Omega^1_X)^*$, therefore the quotient is a complex torus (we will see it is a variety) and it is defined to be *the Albanese variety of* $X$

$$Alb(X) := \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})}$$

Fix a point $x_0 \in X$. The holomorphic map

$$\begin{aligned} \alpha_{x_0} : X &\rightarrow Alb(X) \\ x &\mapsto \{\omega \mapsto \int_{x_0}^{x} \omega\} \end{aligned}$$

is *a canonical map* and the universal property is satisfied.

If $f : X \rightarrow Y$ is a homomorphism between two varieties and $\alpha_Y : Y \rightarrow Alb(Y)$ is a canonical map then the composite induces a homomorphism $v : Alb(X) \rightarrow Alb(Y)$ by means of the universal property.

**• The Picard variety**

If $X$ is a non-singular variety it is defined *the Picard group of* $X$, $Pic(X)$. Its elements are the classes of divisors on $X$ modulo linear equivalence (i.e. isomorphisms).

A *divisor* $D$ on $X$ is *algebraically equivalent to zero* if there exists an irreducible variety $C$ (or an irreducible curve), a divisor $\mathcal{D}$ on $X \times C$ and two points $c, c' \in C$ such that

$$D = \mathcal{D}_c - \mathcal{D}_{c'}$$

where $\mathcal{D}_c = p_1(\mathcal{D} \cdot X \times \{c\})$ and $\mathcal{D}_{c'} = p_1(\mathcal{D} \cdot X \times \{c'\})$.

Two divisors are algebraically equivalent if their difference is algebraically equivalent to zero.

These divisors form a subgroup $Pic^0(X) \subset Pic(X)$ since linearly equivalent divisors have $\mathbb{P}^1$ as parameter variety and thus are also algebraically equivalent

$$Pic^0(X) := \frac{Div_{alg}X}{Div_{lin}X}$$

If in addition $X$ is a non-singular projective complex variety then the group $Pic^0(X)$ admits the structure of an abelian variety and it is called *the Picard variety of* $X$.

In order to show that $Pic^0(X)$ is an abelian variety, first we consider the identification of $Pic(X)$ with $H^1(X, \mathcal{O}_X^*)$ (see for instance [20]).

In terms of holomorphic line bundles, the notion of algebraic equivalence on divisors expresses as follows: a *line bundle* $L$ on $X$ is *algebraically equivalent to zero* if there exist an irreducible parameter variety $T$, a line bundle $\mathcal{L}$ on $X \times T$ and two points $c, c' \in T$ such that

$$\mathcal{L}_{|X \times \{c\}} \cong \mathcal{O}_X$$

and

$$\mathcal{L}_{|X \times \{c'\}} \cong L$$

By connecting $c$ and $c'$ we can reduce to the case of a parameter space which is an irreducible curve $C$.

Let now $T$ denote the holomorphic line bundles on $X$ with first Chern class zero

$$T = Ker\{c_1 : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})\}$$

The exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \longrightarrow 0$$

determines the exact sequence

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})$$

Therefore $T$ can be seen as

$$T \cong H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \cong \overline{H^0(X,\Omega_X^1)}/H^1(X,\mathbb{Z})$$

where the last isomorphism is due to the Hodge decomposition of $H^1(X,\mathbb{C})$

$$\begin{aligned} H^1(X,\mathbb{C}) &\cong H^{1,0}(X) \oplus H^{0,1}(X) = H^{1,0}(X) \oplus \overline{H^{0,1}(X)} \\ &\cong H^0(X,\Omega_X^1) \oplus H^1(X,\mathcal{O}_X) \cong H^0(X,\Omega_X^1) \oplus \overline{H^0(X,\Omega_X^1)} \end{aligned}$$

• *Poincaré divisors.* It is defined a holomorphic line bundle $\mathcal{P}$ on $X \times T$ which allows to show that $T$ and $Pic^0(X)$ are isomorphic. Such a line bundle $\mathcal{P}$ is called *the Poincaré line bundle*.

In order to construct $\mathcal{P}$ let us consider

$$Id \in \ Hom(H^1(X,\mathbb{Z}), H^1(X,\mathbb{Z})) \cong H^1(X,\mathbb{Z}) \otimes H^1(X,\mathbb{Z})^*$$

Since $T = H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is a complex torus, the quotient map

$$\pi : H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) = T$$

is the universal covering map of $T$ thus $Ker(\pi) = H^1(X,\mathbb{Z}) \cong \pi_1(T)$. The fundamental group of $T$, $\pi_1(T)$, is abelian therefore $\pi_1(T) \cong H_1(T,\mathbb{Z})$. It follows that

$$H^1(X,\mathbb{Z}) \cong H_1(T,\mathbb{Z})$$

and consequently $H^1(X,\mathbb{Z})^* \cong H^1(T,\mathbb{Z})$.

Therefore the identity map on $H^1(X,\mathbb{Z})$, $Id$, can be seen as an element of $H^1(X,\mathbb{Z}) \otimes H^1(T,\mathbb{Z})$. By the Künneth decomposition of $H^2(X \times T,\mathbb{Z})$ we can think of $H^1(X,\mathbb{Z}) \otimes H^1(T,\mathbb{Z})$ as a subgroup of $H^2(X \times T,\mathbb{Z})$ and in this manner the element $Id$ determines a homology class on $X \times T$

$$Id \mapsto e \in H^2(X \times T,\mathbb{Z})$$

It can be shown that $e$ is of type $(1,1)$ in the Hodge decomposition of $H^2(X \times T,\mathbb{C})$ (see [19] or [36]).

By Lefschetz theorem on $(1,1)$-cycles there exists a line bundle

$$\mathcal{P} \in H^1(X \times T, \mathcal{O}_{X\times T}^*)$$

defined up to line bundles with vanishing Chern class and such that $c_1(\mathcal{P}) = e$.

The restriction $\mathcal{P}_{|X\times\{\xi\}}$ is a line bundle on $X$, by normalization, i.e. by imposing $\mathcal{P}_{|X\times\{0\}} \cong \mathcal{O}_X$, if $\xi \in T$ represents a line bundle $P_\xi \to X$ then

$$\mathcal{P}_{|X\times\{\xi\}} \cong P_\xi \tag{1.8}$$

If in addition we normalize with respect to a point of $X$, i.e. we require that

$$\mathcal{P}_{|\{x\}\times T} \cong \mathcal{O}_T$$

for a fixed point $x \in X$, then the Poincaré line bundle is unique as we can deduce by the Seesaw Principle (see Theorem 6 in Appendix,§2, of [28] or Corollary $A.9$ of [29])

**Theorem 1.3.1.** *(Seesaw Principle) If $X$ and $Y$ are compact complex manifolds and $\mathcal{L}$ is a holomorphic line bundle on $X \times Y$ such that*

*i) $\mathcal{L}_{|X\times\{y\}}$ is trivial for all $y$ out of an open dense subset of $Y$*

*ii) $\mathcal{L}_{|\{x_0\}\times Y}$ is trivial for some $x_0 \in X$*

*then $\mathcal{L}$ is trivial.*

Indeed if $\mathcal{P}'$ is another holomorphic line bundle on $X \times T$ such that $\mathcal{P}'_{|X\times\{\xi\}} \cong P_\xi$ and $\mathcal{P}'_{|\{x\}\times T} \cong \mathcal{O}_T$ then the theorem implies that $\mathcal{P} = \mathcal{P}'$.

The class of a *Poincaré divisor* corresponding to a Poincaré line bundle is then uniquely determined up to trivial correspondences

$$X \times D_1 + D_2 \times T$$

where $D_1$ is a divisor of $T$ and $D_2$ is a divisor of $X$ (see Chapter $IV$, §4 of [28]).

By means of the Poincaré line bundle $\mathcal{P}$ we are able to determine an isomorphism between $T$ and $Pic^0(X)$. Define the map

$$E : T \to T : t \mapsto P_t = \mathcal{P}_{|X\times\{t\}}$$

This map is well defined because by (1.8) it follows that $P_t$ is algebraically equivalent to zero, thus $E(T) \subseteq Pic^0(X)$. Besides if two line bundles are algebraically equivalent then they have the same first Chern class because they are also topologically equivalent. Therefore $E(T) \subseteq Pic^0(X) \subseteq T$.

It can be shown that $E$ is the identity map (see [36]) obtaining the required isomorphism

$$Pic^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$$

We now give the construction of a polarization on $Pic^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ which makes it an abelian variety.

Let $\alpha \in H^2(X, \mathbb{Z})$ be the first Chern class of an ample line bundle on $X$ and define the bilinear form

$$\begin{aligned} \omega : H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) &\to \mathbb{Z} \\ (\varphi, \psi) \qquad\qquad &\mapsto -\int_X \alpha^{n-1} \wedge \varphi \wedge \psi \end{aligned}$$

Since $T = Pic^0(X)$ is a complex torus, $H^2(T,\mathbb{Z}) = \bigwedge^2 H^1(T,\mathbb{Z}) \cong \bigwedge^2 H^1(X,\mathbb{Z})^*$ and therefore $\omega \in H^2(T,\mathbb{Z})$. The class $\omega$ is the first Chern class of a line bundle if it is of type $(1,1)$ in the Hodge decomposition of $H^2(T,\mathbb{C})$. The $(1,0)$-forms on $T$ identify to linear forms on $H^1(X,\mathbb{C})$ which vanish on $H^{1,0}(X)$ and the $(0,1)$-forms on $T$ identify to linear forms on $H^1(X,\mathbb{C})$ which vanish on $H^{0,1}(X)$, therefore $\omega$ is of type $(1,1)$ if and only if it vanishes on $H^1(X,\mathcal{O}_X) \cong \overline{H^0(X,\Omega^1)}$. Because $\alpha^{n-1} \wedge \varphi^{0,1} \wedge \psi^{0,1} = 0$, the extension of $\omega$ to $H^1(X,\mathbb{C})$ vanishes on $H^1(X,\mathcal{O}_X)$ and hence $\omega$ is of type $(1,1)$. The positivity condition reduces to prove that the form

$$h(\varphi^{0,1},\varphi^{0,1}) := \frac{i}{2}\omega(\varphi^{0,1},\overline{\varphi^{0,1}})$$

is positive definite. It is a consequence of the Hodge-Riemann bilinear relations

**Theorem 1.3.2.** *(Hodge-Riemann bilinear relations) If $\omega$ is the Kähler form on a compact Kähler manifold $X$ of dimension $n$ we define a bilinear form*

$$Q : H^k(X,\mathbb{C}) \otimes H^k(X,\mathbb{C}) \to \mathbb{C} : (\varphi,\psi) \mapsto \int_X \varphi \wedge \psi \wedge \omega^{n-k}$$

*Then*

*i)* $Q(\varphi,\psi) = 0$ *if* $\varphi \in H^{p,q}(X)$, $\psi \in H^{p',q'}(X)$ *and* $(p,q) \neq (q',p')$*;*

*ii)* $i^{p-q}(-1)^{k(k-1)/2}Q(\varphi,\overline{\varphi}) > 0$ *for* $\varphi \in H^{p,q}(X)$, $\varphi \neq 0$ *and primitive.*

Applying the theorem to $\varphi^{1,0} \in H^{1,0}(X)$ we find that

$$h(\varphi^{0,1},\varphi^{0,1}) = \frac{i}{2}\int_X \alpha^{n-1} \wedge \varphi^{1,0} \wedge \overline{\varphi^{1,0}} > 0$$

Therefore $\omega$ is a polarization of $Pic^0(X)$ and hence the Picard variety is an abelian variety.

Moreover there are identifications with dual complex tori

$$\widehat{Alb(X)} \cong Pic^0(Alb(X)) = Pic^0(X)$$

and

$$\widehat{\widehat{Alb(X)}} = Alb(X) \cong \widehat{Pic^0(X)}$$

Since a polarization on $Pic^0(X)$ induces an isogeny, i.e. a surjective morphism with finite kernel,

$$\phi : Pic^0(X) \to Alb(X) \cong \widehat{Pic^0(X)}$$

it follows that also $Alb(X)$ is an abelian variety (see [29]).

- **About homomorphisms between $Alb(X)$ and $Pic^0(X)$**

A homomorphism between two complex tori $A = V/\Lambda$ and $B = V'/\Lambda'$ is a holomorphic map, compatible with the group structures.

If $f : A \to B$ is a homomorphism, it is defined the dual homomorphism between the dual varieties $\hat{f} : \hat{B} \to \hat{A}$ (see [29]).

The isogenies are a special class of homomorphisms between complex tori. An *isogeny* is a surjective homomorphism with finite kernel. Every surjective homomorphism of complex tori factorizes into a surjective homomorphism with a complex torus as kernel and an isogeny (the Stein factorization).

For every isogeny $f : A \to B$ there exists an isogeny $g : B \to A$ such that $gf = \deg(f)Id_A$ and $fg = \deg(f)Id_B$, $1/\deg(f)g$ is the inverse isogeny of $f$ in $Hom_{\mathbb{Q}}(B, A)$ (see [28]). Therefore isogenies define an equivalence relation on the set of complex tori.

For each couple of non-singular projective complex varieties $X$ and $Y$ it is possible to associate to a rational class divisor on $X \times Y$ a homomorphism from $Alb(X)$ to $Pic^0(Y)$ and this map is an isomorphism if we consider divisors modulo trivial correspondences

$$\frac{A^1(X \times Y)}{p_1^* A^1(X) + p_2^* A^1(Y)} \cong Hom(Alb(X), Pic^0(Y))$$

• *Construction of the isomorphism.* Let now $T$ denote the rational class of a divisor in $X \times Y$. Fix a point $x_0 \in X$, let $\alpha_{x_0} : X \to Alb(X)$ be the corresponding Albanese map and let $\alpha$ be the Albanese map on $X \times X$

$$\begin{aligned} \alpha : & X \times X \to Alb(X) \\ & (x_1, x_2) \mapsto \alpha_{x_0}(x_1) - \alpha_{x_0}(x_2) \end{aligned}$$

(indeed it is independent of the choice of the point $x_0$).

Let us define the homomorphism

$$\begin{aligned} h_T : X \times X &\to Pic^0(Y) \\ (x_1, x_2) &\mapsto T(x_1) - T(x_2) \end{aligned}$$

where $T(x) = p_2(T \cdot \{x\} \times Y)$, for a point $x \in X$. In terms of line bundles, if $\mathcal{T}$ represents $T$ then $h_T(x_1, x_2) = \mathcal{T}(x_1) \otimes \mathcal{T}(x_2)^{-1}$ where $\mathcal{T}(x) = \mathcal{T}_{|\{x\} \times Y}$, for $x \in X$. Since $h_T(x_0, x_0) = 0$ there exist unique homomorphisms $h_1, h_2 : X \to Pic^0(Y)$ such that $h_T(x_1, x_2) = h_1(x_1) + h_2(x_2)$, with $h_1(x_0) = h_2(x_0) = 0$ (see Corollary 4, 9.3 in [29]). Because $h_T(x, x) = 0$ for each $x \in X$, it follows that $h_1 = -h_2$ and hence $h_T(x_1, x_2) = h_1(x_1) - h_1(x_2)$.

By the universal property of the Albanese variety, there exists a unique homomorphism $\lambda_T : Alb(X) \to Pic^0(Y)$ such that $h_1 = \lambda_T \circ \alpha_{x_0} + k$, where $k$ is the value $h_1(x_0)$.

Therefore $h_T(x_1, x_2) = \lambda_T \circ \alpha_{x_0}(x_1) - \lambda_T \circ \alpha_{x_0}(x_2)$.

Then the associated map to $T$ is

$$\lambda_T : Alb(X) \to Pic^0(Y)$$

If $T$ is a trivial correspondence, i.e. it is of the type $X_1 \times Y$ or $X \times Y_1$ for $X_1$ divisor in $X$, $Y_1$ divisor in $Y$, then $h_T = 0$ and hence $\lambda_T = 0$. Indeed $(X_1 \times Y)(x)$ and $(X \times Y_1)(x)$ are constant, equivalently the line bundles $(L \oplus \mathcal{O}_Y)(x) = \mathcal{O}_Y$ and $(\mathcal{O}_X \oplus L')(x) = L'$ are constant. Therefore the map

$$\begin{array}{cccc} \Omega : & \dfrac{A^1(X \times Y)}{p_1^* A^1(X) + p_2^* A^1(Y)} & \longrightarrow & Hom(Alb(X), Pic^0(Y)) \\ & T & \longrightarrow & \lambda_T \end{array}$$

is well-defined. The map $\Omega$ is actually an isomorphism (see Chapter VI, §2, Theorem 2 of [28]).

The inverse association is given by the following construction (see [28] and Appendix, §2 of [22]). Let $\lambda : Alb(X) \to Pic^0(Y)$ be a homomorphism. Let us take a Poincaré divisor $E \subset Y \times Pic^0(Y)$ and consider the composed map

$$X \times Y \xrightarrow{\alpha_x \times 1} Alb(X) \times Y \xrightarrow{\lambda \times 1} Pic^0(Y) \times Y$$

where $\alpha_x : X \to Alb(X)$ is a canonical map. Let us define $T$ to be the pull-back of the transpose of the Poincaré divisor $E$

$$T := (\alpha_x \times 1)^* (\lambda \times 1)^* \, ({}^t E)$$

Since $E$ is defined up to trivial correspondences the same property holds for $T$.

## 1.4 The Fulton-MacPherson compactification

In this section we introduce the so called Fulton-MacPherson compactification and state some properties of this space. For further details we refer to their paper [13], which is plenty of ideas and suggestions for further work.

Let $X$ be a non-singular algebraic variety on a field $k$ of dimension $m$. The configuration of $n$ distinct labeled points on $X$ is defined to be

$$F(X, n) = X^n \setminus \bigcup_{a,b} \Delta_{\{a,b\}}$$

It is the complement in the cartesian product of the large diagonals $\Delta_{\{a,b\}}$ where the points with the labels $a$ and $b$ coincide. For each subset $S$ of $\{1, \ldots, n\}$ with at least two elements let $Bl_\Delta(X^S)$ denote the blow-up of the corresponding cartesian product $X^S$ along its diagonal. There is a natural embedding

$$F(X, n) \subset X^n \times \prod_{|S| \geq 2} Bl_\Delta(X^S)$$

and $X[n]$ is defined to be the closure of the configuration space in this product. Because of the deep study of this space by Fulton and MacPherson, it is common today to call it *the Fulton-MacPherson compactification*. Indeed they give an equivalent definition of that space which is more suitable for deriving its properties.

Their idea is to blow-up all the diagonals of $X^n$ with respect to a precise order. The more natural one, would be to blow-up the diagonals starting with the smallest one (with $n$ labels that are equal) and decreasing the number of labels till $\Delta_{\{a,b\}}$. In fact this procedure leads to another compactification which Ulyanov studies in [35] and indicates with $X < n >$ and which in some sense is bigger than $X[n]$.

### • Some properties

The following statements are taken from the Introduction and §3 of [13], we refer to §4 for the proofs.

**Theorem 1.4.1.** *1. $X[n]$ is an irreducible non-singular variety;*

*2. the canonical map $X[n] \to X^n$ is proper;*

*3. the symmetric group $S_n$ acts on $X[n]$ and the isotropy group of any point in $X[n]$ is a solvable group.*

**Theorem 1.4.2.** *For $n \geq 2$, $X[n]$ is the closure of $F(X, n)$ in the product of those $Bl_\Delta(X^S)$ for $S \subset \{1, \ldots, n\}$ with $|S| = 2, 3$.*

**Theorem 1.4.3.** *For each $S \subset N = \{1, \ldots, n\}$ with at least two elements there is a non-singular divisor $D(S) \subset X[n]$, such that:*

*1. the union of these divisors is $X[n] \setminus F(X, n)$;*

*2. any set of these divisors meets transversally, an intersection $D(S_1) \cap \cdots \cap D(S_r)$ is non-empty exactly when each pair $S_i$ and $S_j$ is either disjoint or one is contained in the other;*

*3. by the map $X[n] \to X^S$, the inverse image of the small diagonal is scheme-theoretic the union of the divisors $D(T)$ as $T$ varies over the subsets containing $S$.*

### • Inductive construction by blowing-up

For the Fulton-MacPherson construction we follow §3 of [13]. The definition proceeds by induction on $n$.

By definition $X[1] = X$ and $X[2] = Bl_\Delta X^2$. The exceptional divisor is $D(\{1, 2\})$, there is a proper map $X[2] \to X^2$ and Theorem (1.4.3) holds.

Assume that $X[n]$ has been constructed and define $X[n+1]$. By induction $X[n]$ verifies the properties of the Theorem (1.4.3).

Set $Y_0 = X[n] \times X$ and construct a sequence of blow-ups of non-singular varieties along non-singular subvarieties

$$X[n+1] = Y_n \xrightarrow{\pi_{n-1}} Y_{n-1} \to \cdots \to Y_{k+1} \xrightarrow{\pi_k} Y_k \to \cdots \to Y_1 \xrightarrow{\pi_0} Y_0 = X[n] \times X$$

The map $\pi_k : Y_{k+1} \longrightarrow Y_k$ is the blow-up along the disjoint union of the subvarieties $Y_k U^+$, where $U$ varies over all subsets of $N = \{1, \ldots, n\}$ of cardinality $n-k$.

The varieties $Y_k U^+$ are defined by induction on $k$. With $S$ we indicate a subset of $N$, with $S^+$ the set $S \cup \{n+1\}$ and with $S'$ a subset of $N^+ = N \cup \{n+1\}$. If $k=0$, the varieties $Y_0 S' \subset Y_0$ are defined by:

1. if $S' = S \subset N$ and $|S| \geq 2$ then $Y_0 S' = D(S) \times X$;
2. if $S' = \{a^+\}, a \in N$ then $Y_0 S'$ is the graph of the embedding $X[n] \hookrightarrow X[n] \times X$, induced by the $a$-th projection $X[n] \to X^n \to X$;
3. if $S' = S^+, |S| \geq 2$ then the divisor $D(S) \subset X[n]$ projects to the diagonal $\Delta \subset X^S$ and, with $\Delta$ identified with $X$, $Y_0 S^+$ is the image of the embedding $D(S) \hookrightarrow X[n] \times X$.

Finally $Y_{k+1}(S')$ is the strict transform of $Y_k(S')$ through $Y_{k+1} \to Y_k$, except when $S' = U^+$ and $|U| = n-k$, in this case $Y_{k+1}(S')$ is the exceptional divisor of $\pi_k : Y_{k+1} \to Y_k$.

It follows then that the centers of these blow-ups have codimension $m$ in the case $Y_0\{a^+\}$ inside $Y_{n-1}$, $m+1$ in the other cases:

$$Y_n^{(n+1)} \xrightarrow{\pi_{n-1}} Y_{n-1}^{(n+1)} \to \cdots \to Y_{k+1}^{(n+1)} \xrightarrow{\pi_k} Y_k^{(n+1)} \to \cdots \to Y_1^{(n+1)} \xrightarrow{\pi_0} Y_0^{(n+1)} = X[n] \times X$$

**Notation 1.4.1.** In the sequel we will write $Y_k^{(n)}$ (resp. $D^{(n)}(S)$) instead of $Y_k$ (resp. $D(S)$) when we mean that $Y_k^{(n)}$ appears in the sequence of blow-ups from $X[n-1] \times X$ to $X[n]$ (resp. when $D^{(n)}(S)$ is a divisor in $X[n]$ and $S \subset N$, $S \not\subset N \setminus \{n\}$). When there is no ambiguity or no need we will avoid this heavy notation.

# Chapter 2

# On the intersection motive of a singular threefold

Let $Y$ be a projective complex algebraic variety of dimension three with an isolated singular point $y$. Let us assume that $f : X \to Y$ is a resolution of $Y$ with $D = f^{-1}(y)$ a non-singular surface, i.e. $X$ is a nonsingular projective 3-dimensional variety and $f_{|X \setminus D} : X \setminus D \to Y \setminus \{y\}$ is an isomorphism. The case of several isolated singular points is entirely similar.

According to the topological decomposition theorem (see Beilinson, Bernstein and Deligne [1]) applied to $f : X \to Y$ (see Corti and Hanamura, §1.1 and Theorem 5.7 of [5])

$$Rf_*\mathbb{Q}_X = V \oplus \mathcal{IC}_Y$$

where $V$ is a sheaf supported on the singular point and $\mathcal{IC}_Y$ is the intersection complex of $Y$ (its hypercohomology groups are the intersection cohomology groups of $Y$, $IH^i(Y, \mathbb{Q})$).

Since the intersection cohomology groups $IH^i(Y)$ carry a natural pure Hodge structure (see de Cataldo and Migliorini [8]), it is conjectured that the structure of the intersection cohomology arises as the cohomology of a pure motive (see [5]).

In this chapter we find a motive which realizes the intersection cohomology of $Y$.

In order to do this, first we recall (see [8]) that the intersection cohomology groups of $Y$ are direct summands of the cohomology of the resolution $X$ (see formulae (2.3)), then we find projectors on $X$ which induce the decompositions. In this way each group $IH^i(Y, \mathbb{Q})$ is the cohomology group of a motive and from this viewpoint we can therefore state that the intersection motive of $Y$ is a direct summand of the motive $(X, \Delta)$ of the desingularization.

## 2.1 Motivation

• **The basic setting.** Let $Y$ be a projective complex variety of dimension 3 with an isolated singular point $y$. Let $f : X \to Y$ be a resolution of $Y$, therefore $X$ is a non-singular projective complex variety of dimension 3 and $f$ is an isomorphism when restricted to $X \setminus f^{-1}(y)$. Let us suppose that $f^{-1}(y) = D$ is a smooth surface. Let us consider an ample line bundle $\eta$ on $X$ and an ample line bundle $\mathcal{O}_Y(1)$ on $Y$ and denote with $L$ the pull-back of $\mathcal{O}_Y(1)$ on $X$.

Throughout the chapter we identify (co)homological groups by means of Poincaré duality.

We observe that by the Hard Lefschetz Theorem (see for instance [19]) the cup-product with powers of $\eta$ gives isomorphisms between rational cohomology groups of $X$

$$c_1(\eta)^k : H^{3-k}(X) \to H^{3+k}(X)$$

for all $k \geq 0$.

- By following a procedure in [8], we now want to investigate how the cup product with the non ample line bundle $L$

$$c_1(L)^k : H^{3-k}(X) \to H^{3+k}(X)$$

acts on the cohomology groups of $X$.

If $k = 0$ the corresponding map is the identity. If $k = 3$ then $c_1(L)^3 : H^0(X) \to H^6(X)$ is an isomorphism, indeed $c_1(L)^3 \neq 0$ and both $H^0(X)$ and $H^6(X)$ are generated by one class. Let us then study the cases for $k = 1, 2$.

**Remark 2.1.1.** Let us observe that if $D_i$ are the irreducible components of $D$ then each $D_i$ determines a class in $H_4(X) = H^2(X)$ and we can identify $H_4(D)$ with a subspace of $H^2(X)$.

**Theorem 2.1.1.** *The cup-products with $c_1(L)$ and $c_1(L)^2$ give isomorphisms as follows*

$$c_1(L) : \frac{H^2(X)}{Im\{i_* : H_4(D) \to H^2(X)\}} \to Ker\{i^* : H^4(X) \to H^4(D)\}$$

$$c_1(L)^2 : H^1(X) \to H^5(X)$$

*where $i : D \hookrightarrow X$ is the inclusion.*

In order to prove the above theorem we need the following result (see Proposition 8.2.6 in [10])

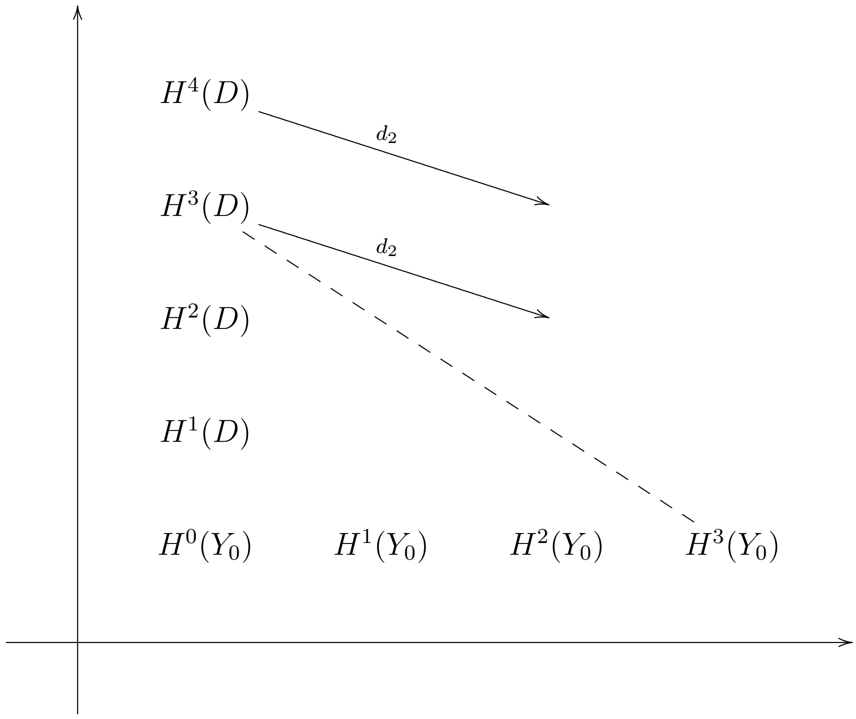
**Proposition 2.1.1.** *If $T$ is a compact algebraic subvariety of a non-singular variety $Z$ and $\overline{Z}$ is a compactification then $H^*(\overline{Z}, \mathbb{Q})$ and $H^*(Z, \mathbb{Q})$ have the same image in $H^*(T, \mathbb{Q})$.*

*Proof of the Theorem.* Let $s \in \Gamma(Y, \mathcal{O}_Y(1))$ be a generic section with smooth zero locus and define $X_s = f^{-1}(\{s = 0\}) \overset{j}{\hookrightarrow} X$.

Let us consider $X_0 = X \setminus X_s$ and $Y_0 = Y \setminus \{s = 0\}$ and study the Leray spectral sequence of $f : X_0 \to Y_0$ (see for instance [19]) for which

$$\begin{aligned} &E_\infty \Longrightarrow H^*(X_0, \mathbb{Q}) \\ &E_2^{p,q} = H^p(Y_0, R^q f_* \mathbb{Q}_{X_0}) \end{aligned}$$

Since $Y_0$ is affine it follows that $H^k(Y_0, \mathbb{Q}) = 0$ if $k > 3$ and because for $q > 0$ the sheaf $R^q f_* \mathbb{Q}_{X_0}$ is supported on the singular point $y$ and $R^q f_* \mathbb{Q}_{X_0|y} = H^q(D, \mathbb{Q})$ we have $E_2^{0,q} \cong H^q(D, \mathbb{Q})$ for $0 < q \leq 4$.




In particular we obtain the exact sequences $H^4(X_0, \mathbb{Q}) \to H^4(D, \mathbb{Q}) \to 0$ and $H^3(X_0, \mathbb{Q}) \to H^3(D, \mathbb{Q}) \to 0$.

By Proposition 2.1.1 the restriction maps $H^4(X, \mathbb{Q}) \to H^4(D, \mathbb{Q})$ and $H^3(X, \mathbb{Q}) \to H^3(D, \mathbb{Q})$ are surjective.

Let us observe that $H^1_c(X_0, \mathbb{Q}) \cong H^5(X_0, \mathbb{Q})^* = 0$, $H^2_c(X_0, \mathbb{Q}) \cong H^4(X_0, \mathbb{Q})^* = H_4(X_0, \mathbb{Q}) = H_4(D, \mathbb{Q})$ and that $H_4(D, \mathbb{Q}) \to H_4(X, \mathbb{Q}) = H^2(X, \mathbb{Q})$ is injective by duality. From the following long exact sequence

$$H^1_c(X_0, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \to H^1(X_s, \mathbb{Q}) \to H^2_c(X_0, \mathbb{Q}) \to H^2(X, \mathbb{Q}) \to H^2(X_s, \mathbb{Q})$$

it follows then that

$$j^* : H^1(X, \mathbb{Q}) \to H^1(X_s, \mathbb{Q})$$

is an isomorphism and that

$$j^* : H^2(X, \mathbb{Q}) / Im\{H_4(D, \mathbb{Q}) \to H^2(X, \mathbb{Q})\} \to H^2(X_s, \mathbb{Q})$$

is injective. By dualizing with respect to Poincaré duality it follows that $j_* : H^3(X_s, \mathbb{Q}) \to H^5(X, \mathbb{Q})$ is an isomorphism and $j_* : H^2(X_s, \mathbb{Q}) \to Ker\{H^4(X, \mathbb{Q}) \to H^4(D, \mathbb{Q})\}$ is surjective. Therefore

$$c_1(L)^2 =: H^1(X, \mathbb{Q}) \to H^1(X_s, \mathbb{Q}) = H^3(X_s, \mathbb{Q}) \to H^5(X, \mathbb{Q})$$

is an isomorphism whereas

$$c_1(L) : H^2(X, \mathbb{Q})/H_4(D, \mathbb{Q}) \to Ker\{H^4(X, \mathbb{Q}) \to H^4(D, \mathbb{Q})\}$$

is an isomorphism if and only if the image of $H^2(X, \mathbb{Q})/H_4(D, \mathbb{Q})$ in $H^2(X_s, \mathbb{Q})$ is mapped isomorphically onto $Ker\{H^4(X, \mathbb{Q}) \to H^4(D, \mathbb{Q})\}$. In fact this is the case because $H^2(X_s, \mathbb{Q})$ satisfies the Hodge-Riemann relations and the Poincaré bilinear form remains non-degenerate when resticted from $H^2(X_s, \mathbb{Q})$ to the image of $H^2(X, \mathbb{Q})/H_4(D, \mathbb{Q})$ which is a substructure of Hodge (for equivalent statements of the Hard Lefschetz Theorem see [27]). □

**Remark 2.1.2.** From the observations we made in the proof of Theorem 2.1.1 we deduce the following exact sequences

$$\begin{aligned} H^3(X, \mathbb{Q}) &\to H^3(D, \mathbb{Q}) \to 0 \\ H^4(X, \mathbb{Q}) &\to H^4(D, \mathbb{Q}) \to 0 \\ 0 \to H_4(D, \mathbb{Q}) &\to H^2(X, \mathbb{Q}) \end{aligned}$$

Hence, by means of Theorem 2.1.1, the cup-product with powers of $c_1(L)$ satisfies the Hard Lefschetz Theorem with respect to subgroups of cohomology groups of $X$ as illustrated below

$$\begin{array}{c} H^0(X) \xrightarrow{\qquad\qquad c_1(L)^3 \qquad\qquad} H^6(X) \\ H^1(X) \xrightarrow{\qquad\quad c_1(L)^2 \qquad\quad} H^5(X) \\ H^2(X)/H_4(D) \xrightarrow{c_1(L)} Ker\{H^4(X) \to H^4(D)\} \\ H^3(X) \end{array} \tag{2.1}$$

Moreover it can be proved (see [8]) that the corresponding primitive cohomology groups

$$\begin{aligned} P^1 &:= Ker\{c_1(L)^3 : H^1(X) \to H^7(X)\} = H^1(X) \\ P^2 &:= Ker\{c_1(L)^2 : H^2(X)/H_4(D) \to H^6(X)\} \\ P^3 &:= Ker\{c_1(L) : H^3(X) \to H^5(X)\} \end{aligned}$$

satisfy the Hodge-Riemann relations.

Therefore the groups in (2.1) behave as if they were cohomology groups. From these results we are led to conjecture the existence of a motive on $X$ which realizes such properties, i.e. such that its cohomology groups are expressed by the subgroups of $H^*(X)$ in (2.1).

In addition, by applying the Decomposition Theorem, such groups can be expressed through the intersection cohomology groups of the singular variety $Y$ (see [8]).

The standard reference for the Decomposition Theorem of Beilinson, Bernstein and Deligne is [1] (specifically Theorem 6.2.5). If $f : X \to Y$ is a projective morphism of complex varieties and $X$ is non-singular then the Theorem can be formulated in the following way (see [5] or Theorem 2.1.1 of [8]):

**Theorem 2.1.2 (The Topological Decomposition Theorem).** *Let $f : X \to Y$ be a map of projective complex varieties and $X$ be non-singular of dimension $n$. Then the direct image sheaf $Rf_*\mathbb{Q}_X[n]$ splits in $D^b_{cc}(Y)$, the bounded derived category of cohomologically constructible sheaves on $Y$, as follows*

$$Rf_*\mathbb{Q}_X[n] \cong \bigoplus_i {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n])[-i]$$

*where ${}^p\mathcal{H}^i$ is the i-th perverse cohomology.*

**Theorem 2.1.3 (The Semisimplicity Theorem).** *In the same hypotesis of the previous theorem, given a stratification for $f$, $Y = \amalg_l S_l$, $0 \leq l \leq \dim(Y)$, there is a canonical isomorphism of perverse sheaves*

$${}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n]) \cong \bigoplus_l IC_{\overline{S_l}}(L_{i,l})$$

*where $IC_{\overline{S_l}}(L_{i,l})$ are the intersection cohomology complex on $Y$ associated with certain local systems defined on $S_l$. When $f$ is semismall the local systems $L_{i,l}$ reduce to $L_{0,l} = R^{n-\dim Y_l} f_*\mathbb{Q}_{X|Y_l}$.*

Let us define the defect of semismallness of $f$

$$r(f) := max_i\{2i + \dim Y^i - \dim X\}$$

where $Y^i = \{y \in Y |\dim f^{-1}(y) = i\}$. If $f$ is semismall, i.e. $r(f) = 0$, then $Rf_*\mathbb{Q}_X[n] \cong {}^p\mathcal{H}^0(Rf_*\mathbb{Q}_X[n])$ and the strata we have to consider are only those for which $2\dim f^{-1}(y) = \dim Y - \dim Y_l$, for all $y \in Y_l$. In general, we have the vanishing of the terms which in the decomposition theorem correspond to indexes $i$ with $|i| > r(f)$ (see [8]).

In the case $f : X \to Y$ is a resolution of the isolated singularity $y$ of $Y$, the stratification consists of two strata $S_3 = Y \setminus \{y\}$ and $S_0 = \{y\}$ and we have $r(f) = 1$. Therefore

$$Rf_*\mathbb{Q}_X[3] \cong {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[3])[1] \oplus {}^p\mathcal{H}^0(Rf_*\mathbb{Q}_X[3])[0] \oplus {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[3])[-1]$$

where ${}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[3])$ and ${}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[3])$ are skyscraper sheaves on $Y$ supported on $y$ and which in addition are isomorphic by means of the Relative Hard Lefschetz Theorem (see Theorem 2.1.1 of [8]): for every $i \geq 0$ the map induced by $\eta$ in perverse cohomology is an isomorphism

$$\eta^i : {}^p\mathcal{H}^{-i}(Rf_*\mathbb{Q}_X[n]) \to {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n])$$

By the axioms which determine the intersection complexes $IC_{\overline{S_l}}(L_{i,l})$ (see §3.8 of [8]) we have the following identifications

$$
\begin{aligned}
&H^3(D,\mathbb{Q}) = H^0(D,\mathbb{Q}[3])\\
&H^4(D,\mathbb{Q}) = H^1(D,\mathbb{Q}[3]) \cong {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[3])_y\\
&H_4(D,\mathbb{Q}) \cong {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[3])_y
\end{aligned}
\tag{2.2}
$$

Hence in particular the Relative Hard Lefschetz Theorem expresses the isomorphism

$$
\begin{aligned}
c_1(\eta) : H_4(D,\mathbb{Q}) &\to H^4(D,\mathbb{Q})\\
[D_i] \qquad &\mapsto \left\{[D_j] \mapsto \int_X c_1(\eta)\wedge[D_i]\wedge[D_j]\right\}
\end{aligned}
$$

where $D_i$ are the components of $D$. Indeed the bilinear form $(D_i, D_j) \mapsto \int_X c_1(\eta)\wedge[D_i]\wedge[D_j]$ is non-degenerate since it is the intersection form of the exceptional curves $D_i \cap H$ of the restriction $f_{|H}$, where $H$ is a $\eta$-hyperplane section of $X$, and we have the classical result of Grauert on contractible curves on surfaces.

Finally we observe the properties noted at Remark 2.1.2 and that the intersection cohomology groups of $Y$ are the hypercohomology groups of the intersection complex $IC_Y$

$$IH^{n+i}(Y,\mathbb{Q}) \cong \mathbb{H}^i(Y, IC_Y(\mathbb{Q}_U))$$

where $U$ is the open stratum. The degeneration of the perverse Leray spectral sequence of $f$

$$\mathbb{H}^i(Y,\ {}^p\mathcal{H}^j(Rf_*\mathbb{Q}_X)) \Longrightarrow H^{i+j+n}(X,\mathbb{Q})$$

then implies the following statement (see [8]):

**Theorem 2.1.4.** *Let $f: X \to Y$ be a resolution of the singularity of $Y$ as above. Then the intersection cohomology groups of $Y$ are direct summands of the cohomology groups of $X$ and the following cohomological decompositions hold*

$$
\begin{aligned}
&H^0(X) \cong IH^0(Y)\\
&H^1(X) \cong IH^1(Y)\\
&H^2(X) \cong H_4(D) \oplus IH^2(Y)\\
&H^3(X) \cong H^3(D) \oplus IH^3(Y)\\
&H^4(X) \cong H^4(D) \oplus IH^4(Y)\\
&H^5(X) \cong IH^5(Y)\\
&H^6(X) \cong IH^6(Y)
\end{aligned}
\tag{2.3}
$$

In the next sections of this chapter we construct a motive whose cohomology turns out to be the intersection cohomology of $Y$, $IH^*(Y)$. In addition, we obtain that this intersection motive is a direct summand of the motive of the desingularization $X$.

## 2.2 The trivial, the Albanese, and the Picard motives

Let $f : X \to Y$ be a resolution of the singularity $y$ of $Y$ as in the basic setting.

In this section we analyse the intersection cohomology groups $IH^i(Y)$, with $i = 0, 1, 5, 6$.

From Theorem 2.1.4 we observe that the intersection cohomology groups $IH^0(Y)$, $IH^1(Y)$, $IH^5(Y)$ and $IH^6(Y)$ are isomorphic to the corresponding cohomology groups of $X$. In order to realize such groups, we recall the construction of the algebraic projectors on $X$ which realize $H^0(X)$, $H^1(X)$, $H^5(X)$ and $H^6(X)$ respectively and for the well-known results we principally refer to the papers of Kleiman [22], Murre [30] and Scholl [34].

**• The trivial projectors**

According to the construction of Murre [30], let $e$ be a point on $X$. Let us consider the two correspondences

$$e \times X \in A^3(X \times X)$$
$$X \times e \in A^3(X \times X)$$

**Proposition 2.2.1.** *For a fixed point $e$ of $X$, the correspondences $e \times X$ and $X \times e$ in $A^3(X \times X)$ are Chow projectors and define the trivial Chow motives $(X, e \times X)$ and $(X, X \times e)$.*

*Proof.* The composite correspondence $(e \times X)^2$ is the rational class

$$\begin{aligned}(e \times X) \circ (e \times X) &= p_{13*}(p_{12}^*(e \times X) \cdot p_{23}^*(e \times X)) \\ &= p_{13*}((e \times X \times X) \cdot (X \times e \times X)) \\ &= (e \times X)\end{aligned}$$

and analogously for $(X \times e)^2$

$$\begin{aligned}(X \times e) \circ (X \times e) &= p_{13*}(p_{12}^*(X \times e) \cdot p_{23}^*(X \times e)) \\ &= p_{13*}((X \times e \times X) \cdot (X \times X \times e)) \\ &= (X \times e)\end{aligned}$$

Therefore $e \times X$ and $X \times e$ are Chow projectors and they define the Chow motives $(X, e \times X)$ and $(X, X \times e)$ respectively. □

In order to realize the intersection cohomology groups $IH^0(Y)$ and $IH^6(Y)$ as cohomology groups of a motive on $X$, we now recall how the trivial projectors act on the cohomology of $X$ and hence we determine the cohomology of the motives $(X, e \times X)$ and $(X, X \times e)$.

We also recall that the notation for the Künneth components of the image of the class of the diagonal $\Delta \in A^n(X \times X)$ through the cycle class map, $cl : A^*(X \times X) \to H^*(X \times X)$, are indicated by $\pi_i$, i.e. $cl(\Delta) = \sum_i \pi_i$ with $\pi_i \in H^{2n-i}(X) \otimes H^i(X)$.

**Proposition 2.2.2.** *As homological correspondences, the projector $e \times X$ is the Künneth component of the diagonal $cl(\Delta)$ in $H^6(X) \otimes H^0(X)$ and the projector $X \times e$ is the Künneth component of the diagonal $cl(\Delta)$ in $H^0(X) \otimes H^6(X)$.*

*Proof.* Through the cycle class map $e \times X \in H^6(X) \otimes H^0(X)$ and $X \times e \in H^0(X) \otimes H^6(X)$ define two homological classes in $H^6(X \times X)$ and therefore two homomorphisms

$$(e \times X)_*, (X \times e)_* : H^i(X) \to H^i(X)$$

Let $\alpha \in H^i(X)$ be a cohomology class on $X$, then

$$\begin{aligned}(e \times X)_*(\alpha) &= p_{2*}((e \times X) \cdot p_1^*(\alpha)) \\ &= p_{2*}((e \times X) \cdot (\alpha \times X)) \\ &= deg(e \cdot \alpha) X\end{aligned}$$

The image class vanishes if $i \neq 0$, otherwise $(e \times X)_*$ is the identity, in other words, $(e \times X)_*$ is the projection from $H^*(X)$ to $H^0(X)$, i.e. it corresponds to $\pi_0$, the $(6,0)$-Künneth component of $cl(\Delta)$.

In an analogous way it can be seen that the homomorphism associated with $X \times e$ between cohomology groups is the projection from $H^*(X)$ to $H^6(X)$ and hence it corresponds to $\pi_6$, the $(0,6)$-Künneth component of $cl(\Delta)$. □

**Proposition 2.2.3.** *The intersection cohomology groups $IH^0(Y)$ and $IH^6(Y)$ are the cohomology groups of the Chow motives $(X, e \times X)$ and $(X, X \times e)$ respectively. Moreover such groups are the kernel of the homological homomorphisms induced by the Chow projectors $\Delta - (e \times X)$ and $\Delta - (X \times e)$:*

$$\begin{aligned}IH^0(Y) &= H^*((X, (e \times X))) = H^0((X, (e \times X))) \\ IH^6(Y) &= H^*((X, (X \times e))) = H^6((X, (X \times e)))\end{aligned}$$

*and*

$$IH^0(Y) = Ker(\Delta_* - (e \times X)_*), \qquad IH^6(Y) = Ker(\Delta_* - (X \times e)_*)$$

*Proof.* From the above proposition and the cohomological decomposition on $X$ it follows that

$$\begin{aligned}H^*(X, e \times X) &= cl(e \times X)_* H^*(X) && \text{(by definition)} \\ &= cl(e \times X)_* H^0(X) = \pi_{0*} H^0(X) && \text{(by Proposition 2.2.2)} \\ &= H^0(X) \\ &\cong IH^0(Y) && \text{(by Theorem 2.1.4)}\end{aligned}$$

and analogously

$$\begin{aligned}H^*(X, X \times e) &= cl(X \times e)_* H^*(X) && \text{(by definition)} \\ &= cl(e \times X)_* H^6(X) = \pi_{6*} H^0(X) && \text{(by Proposition 2.2.2)} \\ &= H^6(X) \\ &\cong IH^6(Y) && \text{(by Theorem 2.1.4)}\end{aligned}$$

Since $(e \times X)$ is a projector in $A^3(X \times X)$ it follows that $Id - (e \times X) = \Delta - (e \times X)$ is a Chow projector and in particular $Im((e \times X)_*) = Ker(\Delta_* - (e \times X)_*)$ when we regard $(X, e \times X)$ both as a Chow motive and as a homological motive. In an analogous way we infer the corresponding statement for $(X, X \times e)$. □

**Remark 2.2.1.** The cycle classes of $e \times X$ and $X \times e$ depend on the choice of the point $e$ but the motives $(X, e \times X)$ and $(X, X \times e)$ are unique up to isomorphisms (see [23] or Remark 1.6.4 of [30]).

**• The Picard and the Albanese projectors**

We show that the intersection cohomology groups $IH^1(Y)$ and $IH^5(Y)$ are obtained as the cohomology groups of the Picard and the Albanese motive on $X$ respectively. We recall the construction of these Chow motives according to the paper of Murre [30].

Fix a projective embedding of $X$ in some $\mathbb{P}^N$ and fix a point $e$ on $X$. It is possible to choose a smooth 1-dimensional linear section $C$ on $X$ passing through $e$ (see §2.1 of [30]). The inclusion $i : C \hookrightarrow X$ determines a homomorphism of abelian varieties $\alpha$ which makes the following diagram commutative

$$\begin{array}{ccc} C & \xrightarrow{i} & X \\ {\scriptstyle alb_C}\downarrow & & \downarrow{\scriptstyle alb_X} \\ J(C) & \xrightarrow{\alpha} & Alb(X) \end{array}$$

where $alb_C$ and $alb_X$ are the Albanese maps on $C$ and $X$ respectively, vanishing on $e$. If

$$\beta : Pic^0(X) \longrightarrow \widehat{J(C)}$$

is the dual map of $\alpha$, then it is defined the composite homomorphism

$$\psi = \alpha \circ \beta : Pic^0(X) \longrightarrow Alb(X)$$

where $J(C)$ and its dual variety $\widehat{J(C)}$ have been identified by means of the isomorphism associated to the theta divisor $\Theta$. It can be proven that the map $\psi$ is an isogeny, i.e. it is a surjective homomorphism between complex tori and $Ker(\psi)$ is finite (see Lemma 2.3. ii) of [30]). Moreover it can be shown that $\psi$ is independent of the choice of the curve $C$ (see Lemma 2.3. i) of [30]).

Given an isogeny we have defined the inverse isogeny (see for instance Proposition 2.6 of [29] or page 29 of [28])

$$\varphi : Alb(X) \longrightarrow Pic^0(X)$$

such that $\varphi \circ \psi = n \cdot id_{Pic^0(X)}$ and $\psi \circ \varphi = n \cdot id_{Alb(X)}$, where $n \geq 1$ is the degree of $\psi$ (see §1.3, Chapter 1).

As observed in §1.3, Chapter 1 (see also for instance Theorem 3.9 of [34]) the following isomorphism holds

$$Hom(Alb(X), Pic^0(X)) \otimes \mathbb{Q} \cong \frac{A^1(X \times X)}{p_1^* A^1(X) + p_2^* A^1(X)}$$

and therefore $\varphi$ is associated with the rational class of a divisor $E$ on $X \times X$, defined up to trivial correspondences

$$D_1 \times X + X \times D_2$$

where $D_1$ and $D_2$ are divisors on $X$.

Consider the restriction of $E$ to $C \times X$, $\tilde{E} = E \cdot (C \times X) \in A^1(C \times X)$. As observed in §1.3 of Chapter 3, by normalizing $\tilde{E}$ with respect to the point $e$, it is possible to choose a representative for $\tilde{E}$ such that

$$\begin{aligned}\tilde{E}(e) &= p_{2*}(\tilde{E} \cdot p_1^*(e)) \\ &= p_{2*}(\tilde{E} \cdot (e \times X)) = 0\end{aligned}$$

$$\begin{aligned}{}^t\tilde{E}(e) &= p_{1*}(\tilde{E} \cdot p_2^*(e)) \\ &= p_{1*}(\tilde{E} \cdot (C \times e)) = 0\end{aligned}$$

where $p_1 : C \times X \to C$ and $p_2 : C \times X \to X$ are the projections.

The rational class $\tilde{E}$ on $C \times X$ determines a cycle class in $A^3(X \times X)$ through the map induced by the inclusion $i \times id_X : C \times X \hookrightarrow X \times X$

$$(i \otimes id_X)_* : A^1(C \times X) = A_3(C \times X) \to A_3(X \times X) = A^3(X \times X)$$

We still denote by $\tilde{E}$ the image of $\tilde{E} \in A^1(C \times X)$ in $A^3(X \times X)$.

**Proposition 2.2.4.** *The correspondence $\tilde{E} \in A^3(X \times X)$ determines a projector $\frac{1}{n}\tilde{E} \in A^3(X \times X, \mathbb{Q})$.*

*Proof.* We sketch the proof made by Murre in §3 of [30] to show that $\tilde{E}^2 = n\tilde{E}$ from which it follows that $1/n\tilde{E}$ is a Chow projector. Since $\tilde{E}$ is supported on $C \times X$, the composite correspondence

$$\tilde{E} \circ \tilde{E} = p_{13*}((\tilde{E} \times X) \cdot (X \times \tilde{E})) \in A^3(X \times X)$$

represents the rational class of a divisor supported on $C \times X$. Through the isomorphism between rational divisor classes on a product (up to trivial correspondences) and homomorphisms from the Albanese variety of the first variety to the Picard variety of the second one (see §1.3), the class $\tilde{E}$ corresponds to $\varphi \circ \alpha : J(C) \to Pic^0(X)$ and the class $\tilde{E}_{|C \times C}$ corresponds to $\hat{\alpha} \circ \varphi \circ \alpha : J(C) \to \widehat{J(C)} \cong J(C)$.

If $u - u'$ is a divisor on $C$ then, as homomorphism,

$$\begin{aligned}\tilde{E}^2(u-u') &= \varphi \circ \alpha(\hat{\alpha} \circ \varphi \circ \alpha)(u-u') \\ &= \varphi \circ \psi \circ \varphi \circ \alpha(u-u') \\ &= n\varphi \circ \alpha(u-u') \\ &= n\tilde{E}(u-u')\end{aligned}$$

Therefore $\tilde{E}^2$ and $n\tilde{E}$ correspond to the same rational divisor class in $C \times X$, up to trivial correspondences. By using the normalization for $\tilde{E}$ with respect to the point $e$, we obtain $\tilde{E}^2(e) = 0$ and ${}^t\tilde{E}^2(e) = 0$. Hence $\tilde{E}^2 = n\tilde{E}$ in $A^3(X \times X, \mathbb{Q})$. □

The Chow motive

$$(X, \frac{1}{n}\tilde{E})$$

is called *the Picard motive*. The transpose projector $\frac{1}{n}\,{}^t\tilde{E}$ defines another Chow motive

$$(X, \frac{1}{n}\,{}^t\tilde{E})$$

which is called *the Albanese motive*.

In the paper [30], Murre shows that as cohomology classes the projectors $cl(1/n\tilde{E})$ and $cl(1/n\,{}^t\tilde{E})$ are the $(5,1)$ and the $(1,5)$-Künneth components of $cl(\Delta) \in H^6(X \times X)$ respectively (see §3 and §4 of [30]):

**Proposition 2.2.5.** *The homological projector $cl(\frac{1}{n}\tilde{E})$ operates as zero on $H^i(X)$ if $i \neq 1$ and it is the identity on $H^1(X)$ whereas the homological projector $cl(\frac{1}{n}\,{}^t\tilde{E})$ operates as zero on $H^i(X)$ if $i \neq 5$ and it is the identity on $H^5(X)$.*

We deduce the following fact:

**Proposition 2.2.6.** *The intersection cohomology groups $IH^1(Y)$ and $IH^5(Y)$ are the cohomology groups of the Chow motives $(X, \frac{1}{n}\tilde{E})$ and $(X, \frac{1}{n}\,{}^t\tilde{E})$ respectively. Moreover such groups are the kernel of the homology homomorphisms induced by the Chow projectors $\Delta - (\frac{1}{n}\tilde{E})$ and $\Delta - (\frac{1}{n}\,{}^t\tilde{E})$:*

$$\begin{aligned}IH^1(Y) &= H^*((X, \frac{1}{n}\,\tilde{E})) = H^1((X, \frac{1}{n}\,\tilde{E})) \\ IH^5(Y) &= H^*((X, \frac{1}{n}\,{}^t\tilde{E})) = H^5((X, \frac{1}{n}\,{}^t\tilde{E}))\end{aligned}$$

*and*

$$IH^1(Y) = Ker(\Delta_* - (\frac{1}{n}\,\tilde{E})_*), \qquad IH^5(Y) = Ker(\Delta_* - (\frac{1}{n}\,{}^t\tilde{E})_*)$$

*Proof.* From the above proposition and the cohomological decomposition on $X$ it follows that for the Picard motive

$$\begin{aligned} H^*(X, 1/n\tilde{E}) &= cl(1/n\tilde{E})_* H^*(X) && \text{(by definition)} \\ &= cl(1/n\tilde{E})_* H^1(X) = \pi_{1*} H^1(X) && \text{(by Proposition 2.2.5)} \\ &= H^1(X) \\ &\cong IH^1(Y) && \text{(by Theorem 2.1.4)} \end{aligned}$$

and analogously for the Albanese motive

$$\begin{aligned} H^*(X, 1/n\,{}^t\tilde{E}) &= cl(1/n\,{}^t\tilde{E})_* H^*(X) && \text{(by definition)} \\ &= cl(1/n\,{}^t\tilde{E})_* H^5(X) = \pi_{5*} H^5(X) && \text{(by Proposition 2.2.5)} \\ &= H^5(X) \\ &\cong IH^5(Y) && \text{(by Theorem 2.1.4)} \end{aligned}$$

The correspondences $\Delta - (1/n\tilde{E})$ and $\Delta - (1/n\,{}^t\tilde{E})$ are Chow projectors and in particular $Im((1/n\tilde{E})_*) = Ker(\Delta_* - (1/n\tilde{E})_*)$ when we regard $(X, 1/n\tilde{E})$ both as a Chow motive and as a homology motive. The corresponding statement for the Albanese motive $(X, 1/n\,{}^t\tilde{E})$ follows analogously. □

## 2.3 Construction of the projections on $H_4(D)$ and $H^4(D)$

In this section we construct two Chow motives which allow us to identify the intersection cohomology groups $IH^2(Y)$ and $IH^4(Y)$ with direct summands of the cohomology of the corresponding Chow motive. We can also think of these groups as the cohomology of two homological motives. In the determination of such motives we make use of the cohomological decomposition of the groups $H^2(X)$ and $H^4(X)$ as we can infer from [8] (see Theorem 2.1.4 in the preliminary section of this chapter).

Specifically, from the decomposition $H^2(X) \cong H_4(D) \oplus IH^2(Y)$ we are led to search for a projector in $A^3(X \times X)$ which induces the projection from $H^*(X)$ to $H_4(D)$ such that $IH^2(Y)$ is a direct summand of the kernel of the projector.

Analogously for $IH^4(Y)$, starting from the splitting $H^4(X) \cong H^4(D) \oplus IH^4(Y)$, we want to construct a Chow projector in $X \times X$ which projects the cohomology of $X$ onto $H^4(D)$ and finally we deduce the corresponding property for $IH^4(Y)$.

Recall that with $cl$ we indicate the cycle class map $cl : A^i(X) \to H^{2i}(X)$. To our purpose let $\eta$ be an ample vector bundle on $X$. We recall the following useful result (see [8]):

**Proposition 2.3.1.** *The cup product with $c_1(\eta)$ provides an isomorphism between $H_4(D)$ and $H^4(D)$.*

*Proof.* The map $c_1(\eta) : H_4(D) \to H^4(D)$ maps the component $D_i$ to the map $(D_j \mapsto \int_X c_1(\eta) \cup D_i \cup D_j)$. It is an isomorphism since, as we recalled in §2.1, the bilinear form

$$\begin{aligned} H_4(D) \times H_4(D) &\to \mathbb{Z} \\ (D_i, D_j) &\mapsto \int_X c_1(\eta) \cup D_i \cup D_j \end{aligned}$$

is negative definite. □

Our first statement is that we can project $H^*(X)$ to $H_4(D)$ through an algebraic cycle in $X \times X$:

**Theorem 2.3.1.** *There exists an algebraic cycle $\gamma_2 \in A_3(X \times X)$ defining a Chow motive $(X, \gamma_2)$ which satisfies the following properties:*

*a) $H^i((X, \gamma_2)) = 0$ if $i \neq 2$;*

*b) $H^2((X, \gamma_2)) = H_4(D, \mathbb{Q})$ and $Ker(\gamma_{2*|H^2(X)}) = (\eta \cdot D)^\perp$.*

*In particular the cohomology group $H^2(X)$ decomposes in the following way with respect to $\gamma_{2*}$*

$$H^2(X, \mathbb{Q}) \cong H_4(D, \mathbb{Q}) \oplus (\eta \cdot D)^\perp \tag{2.4}$$

*Proof.* Define $\overline{\gamma}_2 = (\eta \cdot D) \times D \in A_1(X) \otimes A_2(X)$ where $\eta \cdot D$ is the cap product $c_1(\eta) \cap D$.

Observe that, modulo an integer, $\overline{\gamma}_2$ is a projector if seen as a correspondence from $X$ to $X$. Indeed, if $p_{ij} : X \times X \times X \to X \times X$ are the $ij$-projections, by definition we have

$$\begin{aligned} \overline{\gamma}_2 \circ \overline{\gamma}_2 &= p_{13*}(p_{12}^*\overline{\gamma}_2 \cdot p_{23}^*\overline{\gamma}_2) && \text{(by definition)} \\ &= p_{13*}((\eta \cdot D \times D \times X) \cdot (X \times \eta \cdot D \times D)) \\ &= p_{13*}(\eta \cdot D \times \eta \cdot D \cdot D \times D) \\ &= \deg(\eta \cdot D \cdot D)\ (\eta \cdot D) \times D \\ &= m\overline{\gamma}_2 \end{aligned}$$

where $m = \deg(\eta \cdot D \cdot D)$ is a negative integer since the bilinear form

$$H_4(D) \times H_4(D) \to \mathbb{Z} : (D_i, D_j) \mapsto \int_X c_1(\eta) \cup D_i \cup D_j$$

is negative definite. Indeed it is the intersection form of the exceptional curves of the restriction of $f$ to a $\eta$-hyperplane section of $X$. Therefore we define

$$\gamma_2 := \frac{1}{m}\overline{\gamma}_2 \in A_3(X \times X, \mathbb{Q})$$

and it follows that $\gamma_2 \circ \gamma_2 = \gamma_2$, i.e. $\gamma_2$ is a Chow projector.

The cohomology groups of the associated motive $(X,\gamma_2)$ are by definition the images of the induced map $cl(\gamma_2)_*$ in cohomology

$$H^i((X,\gamma_2)) = cl(\gamma_2)_* H^i(X)$$

Let $\alpha \in H^i(X)$ be a cohomology class on $X$. In order to simplify the notation, for the moment $cl(\gamma_2) = cl(\eta \cdot D) \times cl(D) \in H^*(X \times X)$ is denoted with $\gamma_2 = \eta \cdot D \times D$. Then

$$\begin{aligned} \gamma_{2*}(\alpha) &= p_{2*}(\gamma_2 \cdot p_1^*\alpha) \\ &= m\ p_{2*}((c_1(\eta) \cap D) \times D \cdot (\alpha \times X)) \\ &= m\ \deg((c_1(\eta) \cap D) \cdot \alpha)\ D \end{aligned} \tag{2.5}$$

Regarding $c_1(\eta)$ in $H^2(X)$ and $D$ in $H^2(X)$, it follows that if $i \neq 2$

$$\deg((c_1(\eta) \cap D) \cdot \alpha) = \int_X c_1(\eta) \cup D \cup \alpha = 0$$

and therefore in this case

$$H^i((X,\gamma_2)) = \gamma_{2*} H^i(X) = 0$$

As to $\gamma_{2*} : H^2(X) \to H^2(X)$, from (2.5) it follows that

$$Ker(\gamma_{2*}) = \left\{\alpha \in H^2(X) \middle| \int_X c_1(\eta) \cup D \cup \alpha = 0\right\} = (\eta \cdot D)^\perp$$

i.e. the kernel of the map $\gamma_{2*}$ is the orthogonal space to $\eta \cdot D \in H^4(X)$, regarded as an element of $H^2(X)^*$ by Poincaré duality. Thanks to (2.5) we also deduce that

$$Im(\gamma_{2*}) = H_4(D) = \bigoplus_i H_4(D_i)$$

and therefore we obtain the cohomology of the Chow motive $(X,\gamma_2)$

$$H^*((X,\gamma_2)) = H^2((X,\gamma_2)) = H_4(D)$$

Since $\gamma_2$ is a projector it follows that $\gamma_{2*} \circ \gamma_{2*} = (\gamma_2 \circ \gamma_2)_* = \gamma_{2*}$ and therefore we have the following decomposition of $H^2(X,\mathbb{Q})$ with respect to $\gamma_{2*}$

$$\begin{aligned} H^2(X,\mathbb{Q}) &\cong Im(\gamma_{2*}) \oplus Ker(\gamma_{2*}) \\ &= H_4(D,\mathbb{Q}) \oplus (\eta \cdot D)^\perp \end{aligned}$$

Therefore the projector $\gamma_2$ can be thought as the projection from $H^*(X)$ to $H_4(D)$

$$\gamma_{2*} : H^*(X) \to H_4(D) \subset H^2(X)$$

□

From the above theorem it follows the following property for $IH^2(Y)$:

**Proposition 2.3.2.** *The intersection cohomology group $IH^2(Y)$ is a direct summand of the cohomology of the Chow motive $(X, \Delta - \gamma_2)$. Besides it is the cohomology of the homological motive $(X, \pi_{2*} - cl(\gamma_2)_*)$ where $\pi_2 \in H^4(X) \otimes H^2(X)$ is the $(4,2)$-Künneth component of $cl(\Delta) \in H^6(X \times X)$*

$$IH^2(Y) = H^*(X, \pi_{2*} - cl(\gamma_2)_*)$$

*Proof.* From Theorem 2.3.1 and the decomposition on $H^2(X)$ it follows that

$$IH^2(Y) = (\eta \cdot D)^\perp$$

Since $\gamma_2$ is a Chow projector, also $\Delta - \gamma_2$ is a Chow projector and it follows then that $IH^2(Y)$ is a direct summand of $Ker(\gamma_{2*}) = Im(\Delta_* - \gamma_{2*})$, i.e. of $H^*(X, \Delta - \gamma_2)$.

As to the homological situation

$$\begin{aligned}
H^*(X, \pi_{2*} - cl(\gamma_2)_*) &= (\pi_{2*} - cl(\gamma_2)_*)H^*(X) && \text{(by definition)}\\
&= (\pi_{2*} - cl(\gamma_2)_*)H^2(X) && \text{(by Theorem 2.3.1)}\\
&= Ker(cl(\Delta)_* - (\pi_{2*} - cl(\gamma_2)_*)) && \\
&\cong IH^2(Y) && \text{(by Theorem 2.1.4)}
\end{aligned}$$

□

In an analogous way, we now construct the projection from $H^*(X)$ to $H^4(D)$ by means of an algebraic projector:

**Theorem 2.3.2.** *There exists an algebraic cycle $\gamma_4 \in A_3(X \times X)$ defining a Chow motive $(X, \gamma_4)$ which satisfies the following properties:*

*a) $H^i((X, \gamma_4)) = 0$ if $i \neq 4$;*

*b) $H^4((X, \gamma_4)) = H^4(D, \mathbb{Q})$ and $Ker(\gamma_{4*|H^4(X)}) = [D]^\perp$.*

*In particular the cohomology group $H^4(X)$ decomposes in the following way with respect to $\gamma_{4*}$*

$$H^4(X, \mathbb{Q}) \cong H^4(D, \mathbb{Q}) \oplus [D]^\perp \tag{2.6}$$

*Proof.* Define $\bar{\gamma}_4 = {}^t\overline{\gamma}_2 = D \times (\eta \cdot D) \in A_1(X) \otimes A_2(X)$ where $\eta \cdot D$ means $c_1(\eta) \cap D$ in algebraic theory, $c_1(\eta) \cup D$ in cohomology theory.

Because $\overline{\gamma_2}$ is a projector, modulo an integer, also $\bar{\gamma}_4$ defines a Chow projector if seen as a correspondence from $X$ to $X$. Indeed

$$\bar{\gamma}_4 \circ \bar{\gamma}_4 = {}^t\overline{\gamma}_2 \circ {}^t\overline{\gamma}_2 = {}^t(\overline{\gamma}_2 \circ \overline{\gamma}_2) = {}^t(m\overline{\gamma}_2) = m\bar{\gamma}_4$$

Therefore we define

$$\gamma_4 = \frac{1}{m}\bar{\gamma}_4 \in A_3(X \times X, \mathbb{Q})$$

and it follows that $\gamma_4^2 = \gamma_4$, i.e. $\gamma_4$ is a Chow projector.

The cohomology groups of the associated motive $(X, \gamma_4)$ are the images of the induced map $cl(\gamma_4)_*$:

$$H^i((X, \gamma_4)) = cl(\gamma_4)_* H^i(X)$$

Let $\alpha \in H^i(X)$ be a cohomology class on $X$. As we have done in the case of $\gamma_2$, we temporarily use the same notation for $\gamma_4$ and its homological class $cl(\gamma_4)$. Then we have

$$\begin{aligned} \gamma_{4*}(\alpha) &= p_{2*}(\gamma_4 \cdot p_1^* \alpha) \\ &= m\ p_{2*}((D \times (c_1(\eta) \cap D)) \cdot (\alpha \times X)) \\ &= m\ \deg(D \cdot \alpha)\ (c_1(\eta) \cap D) \end{aligned} \tag{2.7}$$

The cohomology class associated with $D$ defines an element in $H^2(X)$ and therefore, if $i \neq 4$,

$$\deg(D \cdot \alpha) = \int_X D \cup \alpha = 0$$

and in this case

$$H^i((X, \gamma_4)) = \gamma_{4*} H^i(X) = 0$$

As regards $\gamma_{4*} : H^4(X) \to H^4(X)$, from (2.7) we find that

$$Ker(\gamma_{4*}) = \{\alpha \in H^4(X) | \int_X D \cup \alpha = 0\} = [D]^\perp$$

i.e. the kernel of the restriction to $H^4(X)$ of the map $\gamma_{4*}$ is the orthogonal space to $D \in H^2(X)$, thinking of $D$ as an element of $H^4(X)^*$ by Poincaré duality. Thanks to (2.7) we also deduce that $Im(\gamma_{4*})$ is generated by the homological class of $[c_1(\eta) \cap D]$.

Since $\gamma_4$ is a projector it follows that $\gamma_{4*} \circ \gamma_{4*} = (\gamma_4 \circ \gamma_4)_* = \gamma_{4*}$ and therefore $H^4(X)$ decomposes in the following way

$$H^4(X, \mathbb{Q}) \cong \mathbb{Q}[\eta \cdot D] \oplus [D]^\perp$$

Thanks to Proposition 2.3.1, we can see $\eta \cdot D$ as an element in $H^4(D)$, it corresponds to the cup product with $c_1(\eta) \cup D$ and we can identify $\mathbb{Q}[\eta \cdot D]$ with $H^4(D)$. Therefore

$$H^4(X, \mathbb{Q}) \cong H^4(D, \mathbb{Q}) \oplus [D]^\perp$$

and we obtain the cohomology of the Chow motive $(X, \gamma_4)$

$$H^*((X, \gamma_4)) = H^4((X, \gamma_4)) = H^4(D)$$

Hence the projector $\gamma_4$ can be thought as the projection from $H^*(X)$ onto $H^4(D)$. $\square$

By means of this last projector, it follows the required property for $IH^4(Y)$:

**Proposition 2.3.3.** *The intersection cohomology group $IH^4(Y)$ is a direct summand of the cohomology of the Chow motive $(X, \Delta - \gamma_4)$. Besides it is the cohomology of the homological motive $(X, \pi_{4*} - cl(\gamma_4)_*)$ where $\pi_4 \in H^2(X) \otimes H^4(X)$ is the $(2,4)$-Künneth component of $cl(\Delta) \in H^6(X \times X)$:*

$$IH^4(Y) = H^*(X, \pi_{4*} - cl(\gamma_4)_*)$$

*Proof.* From Theorem 2.3.2 and the splitting on $H^4(X)$ it follows that

$$IH^4(Y) = [D]^{\perp}$$

Since $\gamma_4$ is a Chow projector, the same holds for $\Delta - \gamma_4$. Therefore $IH^4(Y)$ is a direct summand of $Ker(\gamma_{4*}) = Im(\Delta_* - \gamma_{4*})$, i.e. of $H^*(X, \Delta - \gamma_4)$.

As regards the homological situation

$$\begin{aligned} H^*(X, \pi_{4*} - cl(\gamma_4)_*) &= (\pi_{4*} - cl(\gamma_4)_*)H^*(X) && \text{(by definition)} \\ &= (\pi_{4*} - cl(\gamma_4)_*)H^4(X) && \text{(by Theorem 2.3.2)} \\ &= Ker(cl(\Delta)_* - (\pi_{4*} - cl(\gamma_4)_*)) && \\ &\cong IH^4(Y) && \text{(by Theorem 2.1.4)} \end{aligned}$$

□

We can observe that the projectors $\gamma_2$ and $\gamma_4$ are orthogonal:

**Proposition 2.3.4.** *The Chow projectors $\gamma_2$ and $\gamma_4 \in A_3(X \times X)$ are mutually orthogonal projectors.*

*Proof.* For dimension reasons, we have that both the composite $(\eta D \times D) \circ (D \times \eta D) = p_{13*}(D \times \eta D \times X \cdot X \times \eta D \times D)$ and the composite $(D \times \eta D) \circ (\eta D \times D) = p_{13*}(\eta D \times D \times X \cdot X \times D \times \eta D)$ vanish. □

## 2.4 Construction of the projection on $H^3(D)$

In this section we find a homological motive whose cohomology is the intersection cohomology group $IH^3(Y)$, provided the dual of the normal bundle to $D$ in $X$, $N_D X^{\vee}$, satisfies the Hard Lefschetz theorem and gives a map from the surface $D$ to a projective space, for example if $N_D X$ is an antiample line bundle.

The starting point for our research is the decomposition of $H^3(X)$ as a direct sum of $H^3(D)$ and $IH^3(Y)$. We then look for a cycle in $A^3(X \times X)$ which is supported on $D \times D$ and which induces the projection from $H^*(X)$ to $H^3(D)$.

Since $H^3(D) = H^{2\dim D - 1}(D)$ and since from the surjectivity of the restriction map $H^3(X) \to H^3(D)$ (see Remark 2.1.2) it follows the inclusion $H^1(D) \hookrightarrow H^3(X)$ by duality, we are led to think of a relation between the required projector and the Albanese motive on $D$. In fact this is what we are going to show.

As we did in the case of $X$, we make use of the general setting for the construction of the Picard and Albanese motives as one can find for example in §2 and §3 of the paper of Murre [30] in the case of Chow motives, or in the Appendix to §2 of the work of Kleiman [22] for homological motives.

Fixing a projective embedding of $D$ in some $\mathbb{P}^N$, a point $e$ on $D$ and a smooth hyperplane section $C$ passing trhough $e$, it is possible to define an isogeny

$$\psi : Pic^0(D) \longrightarrow Alb(D)$$

by means of the Jacobian $J(C)$. Let $\varphi : Alb(D) \longrightarrow Pic^0(D)$ denote the inverse isogeny to $\psi$ such that $\varphi \circ \psi = n \cdot id_{Pic^0(D)}$ and $\psi \circ \varphi = n \cdot id_{Alb(D)}$, where $n \geq 1$ is the degree of $\psi$.

By means of the isomorphism which we have recalled in §3 of Chapter 1

$$Hom(Alb(D), Pic^0(D)) \otimes \mathbb{Q} \cong \frac{A^1(D \times D)}{p_1^* A^1(D) + p_2^* A^1(D)}$$

we have the class of a divisor $E$ on $D \times D$ corresponding to $\varphi$, its rational class is uniquely determined up to trivial correspondences

$$C_1 \times D + D \times C_2$$

where $C_1$ and $C_2$ are divisors on $D$.

By choosing appropriate classes for the curves $C_1$ and $C_2$ we can normalize $E$ with respect to the point $e$, i.e. we can assume

$$E(e) = p_{2*}(E \cdot (e \times D)) = 0$$

and

$$^tE(e) = p_{1*}(E \cdot (D \times e)) = 0$$

where $p_1, p_2 : D \times D \to D$ are the two projections. The cycle class map

$$cl : A^p(D \times D) \longrightarrow H^{2p}(D \times D)$$

defines a cohomological class $cl(E)$ in $H^2(D \times D)$ associated with $E$.

• **Description of the homology class of $E$ on $D \times D$**

Let us show some known properties of $cl(E) \in H^2(D \times D)$ (see Kleiman [22] and Murre [30]).

**Proposition 2.4.1.** *The homology class of the divisor $E \in A^1(D \times D)$ is of type $(1, 1)$ in the Künneth decomposition of $H^2(D \times D)$:*

$$cl(E) \in H^1(D) \otimes H^1(D)$$

*Proof.* Through the Künneth decomposition of $H^2(D \times D)$

$$H^2(D \times D) \;\cong\; \Big(H^2(D) \otimes H^0(D)\Big) \oplus \Big(H^1(D) \otimes H^1(D)\Big) \oplus \Big(H^0(D) \otimes H^2(D)\Big)$$

we can decompose $cl(E)$ into

$$cl(E) = \varepsilon_{20} + \varepsilon_{11} + \varepsilon_{02}$$

where $\varepsilon_{ij} = \varepsilon_{ij}^i \times \varepsilon_{ij}^j \in H^i(D) \otimes H^j(D)$ are the components. As a cohomological class, if $p_1, p_2 : D \times D \to D$ are the projections then

$$\begin{aligned} E(e) &= p_{2*}(E \cdot p_1^*(e)) \\ &= p_{2*}(E \cdot (e \times D)) \\ &= p_{2*}(\varepsilon_{02} \cdot (e \times D)) \\ &= \varepsilon_{02}^2 \end{aligned}$$

because in $H^*(D \times D)$, $E \cdot (e \times D) = \sum_{ij} \deg(\varepsilon_{ij}^i \cdot e)\varepsilon_{ij}^j$ and $\deg(\varepsilon_{ij}^i \cdot e) = 0$ if $i \neq 0$. As we require $E(e) = 0$ it follows that $\varepsilon_{02}^2 = 0$ and consequently $\varepsilon_{02} = \varepsilon_{02}^0 \times \varepsilon_{02}^2 = 0$. In an analogous way from the equality

$$\begin{aligned} {}^tE(e) &= p_{1*}(E \cdot p_2^*(e)) \\ &= p_{1*}(E \cdot (D \times e)) \\ &= p_{1*}(\varepsilon_{20} \cdot (D \times e)) \\ &= \varepsilon_{20}^2 \end{aligned}$$

and ${}^tE(e) = 0$ we infer that $\varepsilon_{20}^2 = 0$ and $\varepsilon_{20} = 0$. Therefore the homology class $cl(E)$ is of type $(1,1)$, i.e. $cl(E) = \varepsilon_{11} \in H^1(D) \otimes H^1(D)$. □

**Remark 2.4.1.** The $(1,1)$-Künneth component of $cl(E)$ is independent of the choice of the representative of the class of the divisor $E$ of $D \times D$ associated with the homomorphism $\varphi : Alb(D) \to Pic^0(D)$ and in particular it is independent of the point $e \in D$ wich gives the normalization of $E$.

Indeed such a rational class $E$ is the transpose of the pull-back of a Poincaré divisor $P_D$ of $D \times Pic^0(D)$

$${}^tE = (Id_D \times alb_D)^*(Id_D \times \varphi)^* P_D$$

and the $(1,1)$-Künneth component of $P_D$ is the same for all Poincaré divisors (see Propositions 2A1, 2A2 of [22]). In particular the $(1,1)$-component of $cl({}^tE)$ in $H^1(D) \otimes H^1(D)$ is algebraic depending on $D$ (see also Corollary 2A10 of [22]).

We now expose another interesting property of the cycle class of $E$. Such a property will be a key point in the proof of our result. The general statement can be found in Kleiman [22].

**Proposition 2.4.2.** *Let $L : H^*(D) \to H^*(D)$ be the operator of degree two defined by $a \mapsto a \cdot C$, where $C$ is a hyperplane section of the surface $D$ and let us suppose that the Hard Lefschetz Theorem holds for $L$, i.e. let $L$ satisfy conditions such that $L^i : H^{2-i}(D) \to H^{2+i}(D)$ is an isomorphism for all $i$. If $E$ is the rational divisor class on $D \times D$ which is constructed according to the method we showed previously starting from $C$, then the homomorphism associated with $cl({}^tE)$*

$$cl({}^tE)_* : H^3(D) \to H^1(D)$$

*is an isomorphism and moreover it is the $n$-multiple of the Hodge operator $\Lambda$, the inverse homomorphism to $L$.*

*Proof.* We recall that the inclusion $j : C \hookrightarrow D$ induces homomorphisms $\alpha : J(C) \to Alb(D)$ and $\beta : Pic^0(D) \to \widehat{J(C)}$. Besides the following diagrams commute

$$\begin{array}{ccc} H^1(D) & \xrightarrow{j^*} & H^1(C) \\ {\scriptstyle alb_D^*}\uparrow & & \uparrow{\scriptstyle alb_C^*} \\ H^1(Alb(D)) & \xrightarrow{\alpha^*} & H^1(J(C)) \end{array} \qquad \begin{array}{ccc} H^1(C) & \xrightarrow{j_*} & H^3(D) \\ {\scriptstyle cl(P_C)_*}\downarrow & & \downarrow{\scriptstyle cl(P_D)_*} \\ H^1(J(C)) & \xrightarrow{\beta^*} & H^1(Pic^0(D)) \end{array}$$

Since $j_* \circ j^* = L$ (by the projection formula), $cl(P_C)_* \circ alb_C^* = Id_{H^1(J(C))}$ and $\beta^* \circ \alpha^* = \psi^*$, the following diagram is commutative

$$\begin{array}{ccc} H^1(D) & \xrightarrow{L} & H^3(D) \\ {\scriptstyle alb_D^*}\uparrow & & \downarrow{\scriptstyle cl(P_D)_*} \\ H^1(Alb(D)) & \xrightarrow{\psi^*} & H^1(Pic^0(D)) \end{array} \tag{2.8}$$

As $\psi$ is an isogeny it follows that $\psi^* : H^1(Alb(D)) \to H^1(Pic^0(D))$ is an isomorphism (see for instance Lemma 2A6 of [22]).

In general $alb_D^*$ is injective and $cl(P_D)_*$ is surjective. Because $H^0(Alb(D), \Omega^1) \cong H^0(D, \Omega^1)$ (see for instance [19]), it follows that

$$\dim H^1(D) = \dim H^1(Alb(D))$$

and therefore the four spaces in the diagram (2.8) have the same dimension and thus $alb_D^*$ and $cl(P_D)_*$ are isomorphisms.

As we observed in the previous Remark for $cl(E)$, also $cl({}^tE)$ is of type $(1,1)$ in the Künneth decomposition of $H^2(D \times D)$ and therefore its associated map from the cohomology of $X$ maps $H^3(D)$ on $H^1(D)$. In addition it is clear (see Remark 2.4.1) that the following diagram commutes

$$\begin{array}{ccc} H^1(D) & \xleftarrow{cl({}^tE)_*} & H^3(D) \\ {\scriptstyle alb_D^*}\uparrow & & \downarrow{\scriptstyle cl(P_D)_*} \\ H^1(Alb(D)) & \xleftarrow{\varphi^*} & H^1(Pic^0(D)) \end{array} \tag{2.9}$$

Hence $cl({}^tE)_* : H^3(D) \to H^1(D)$ is an isomorphism and moreover it is algebraic. If $\Lambda$ is the inverse isomorphism to $L$ then from the diagrams (2.8) and (2.9) it follows that $cl({}^tE)_* = n\Lambda = nL^{-1}$. □

**• Description of the homology class of $E$ on $X \times X$**

We now define a new correspondence on $X \times X$ starting from the class of $E$ on $D \times D$. We then show that it satisfies the required properties.

The inclusion $D \overset{i}{\hookrightarrow} X$ defines push-forward homomorphisms of Chow groups and of homology groups

$$A_k(D \times D) \xrightarrow{(i \times i)_*} A_k(X \times X)$$

for each $k = 0, \ldots, 4$ and

$$H^{8-i}(D \times D) \cong H_i(D \times D) \xrightarrow{(i \times i)_*} H_i(X \times X) \cong H^{12-i}(X \times X)$$

for $i = 0, \ldots, 8$. In particular we can consider the image of the class of $E$ and thus two new classes on $X \times X$ are defined, an algebraic class

$$\overline{E} := (i \times i)_*(E) \in A_3(X \times X)$$

and a homological class

$$cl(\overline{E}) = (i \times i)_*(cl(E)) \in H_6(X \times X)$$

We prove that under suitable hypotheses on the normal bundle to $D$ in $X$, the correspondence $cl({}^t\overline{E})$ is a homological projector.

**Theorem 2.4.1.** *Let us suppose that $N_D X^\vee$ is an ample line bundle on $D$ and let $E$ be the divisor on $D \times D$ obtained from a hyperplane section of $D$ with respect to $N_D X^\vee$ according to the method illustrated previously. Then $cl({}^t\overline{E})$ defines a homological projector as a correspondence from $X$ to $X$, i.e. a projector in $H^*(X \times X) \cong Hom_{\mathbb{Q}}(H^*(X), H^*(X))$.*

*Proof.* Let $\alpha \in H^j(X)$ be a cohomological class on $X$, for $j = 0, \ldots, 6$. By Poincaré duality we can identify $H_6(X \times X)$ with $H^6(X \times X)$ and the class $cl({}^t\overline{E})$ defines the homomorphism

$$\begin{array}{rccl} cl({}^t\overline{E})_* : & H^j(X) & \longrightarrow & H^j(X) \\ & \alpha & \longmapsto & p_{(2X)*}(cl({}^t\overline{E}) \cdot p^*_{(1X)}\alpha) \end{array}$$

where $p_{(1X)}, p_{(2X)} : X \times X \to X$ are the projections from $X \times X$. We can rewrite the image of $\alpha$ as follows

$$\begin{aligned} cl({}^t\overline{E})_*(\alpha) &= p_{(2X)*}(cl({}^t\overline{E}) \cdot p^*_{(1X)}\alpha) && \text{(by definition)} \\ &= p_{(2X)*}((i \times i)_*(cl({}^tE))) \cdot (\alpha \times X)) \\ &= p_{(2X)*}((i \times i)_*(cl({}^tE) \cdot (i^*(\alpha) \times D))) && \text{(by projection formula)} \end{aligned}$$

From the commutativity of the following diagram

$$\begin{array}{ccc} H^*(D \times D) & \xrightarrow{p_{(2D)*}} & H^*(D) \\ {\scriptstyle (i\times i)_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ H^*(X \times X) & \xrightarrow{p_{(2X)*}} & H^*(X) \end{array}$$

it follows that

$$\begin{aligned} cl({}^t\overline{E})_*(\alpha) &= p_{(2X)*}((i \times i)_*(cl({}^tE) \cdot (i^*\alpha \times D))) \\ &= (i_* \circ p_{(2D)*})(cl({}^tE) \cdot (i^*\alpha \times D)) \\ &= i_*(cl({}^tE)_*(i^*\alpha)) \end{aligned} \tag{2.10}$$

and this class vanishes if $\alpha \in H^j(X)$ with $j \neq 3$. Indeed $cl({}^tE) \in H^2(D \times D)$ defines the homomorphism

$$\begin{array}{cccc} cl({}^tE)_*: & H^j(D) & \longrightarrow & H^{j-2}(D) \\ & \beta & \longmapsto & p_{(2D)*}(cl({}^tE) \cdot p^*_{(1D)}\beta) \end{array}$$

and since $i^*\alpha \in H^j(D)$ and $cl({}^tE) \in H^1(D) \otimes H^1(D)$ as observed in Proposition 2.4.1, if we write $cl({}^tE) ={}^t\varepsilon_{11} = \varepsilon''_{11} \times \varepsilon'_{11}$ then

$$cl({}^tE)_*(i^*\alpha) = \deg(i^*\alpha \cup \varepsilon''_{11})\ \varepsilon'_{11}$$

By the self-intersection formula (see for instance Corollary 6.3 of [12])

$$i^*i_*(cl({}^tE)_*(i^*\alpha)) = c_1(N_DX) \cup cl({}^tE)_*(i^*\alpha) \tag{2.11}$$

Since $N_DX^\vee$ is an ample line bundle, the Hard Lefschetz Theorem holds, i.e. $c_1(N_DX^\vee)^i : H^{2-i}(D) \to H^{2+i}(D)$ is an isomorphism for all $i$. If $\Lambda : H^3(D) \to H^1(D)$ is the inverse isomorphism to $c_1(N_DX^\vee)$, then Proposition 2.4.2 implies $cl({}^t\overline{E})_* = n\Lambda$.

Therefore

$$\begin{aligned} (cl({}^t\overline{E})_* \circ cl({}^t\overline{E})_*)(\alpha) &= i_*(cl({}^tE)_*(i^*i_*cl({}^tE)_*(i^*\alpha))) && \text{(by (2.10))} \\ &= i_*(cl({}^tE)_*(c_1(N_DX) \circ cl({}^tE)_*(i^*\alpha))) && \text{(by (2.11))} \\ &= n\ i_*(cl({}^tE)_*(c_1(N_DX) \circ \Lambda(i^*\alpha))) && \text{(by Proposition 2.4.2)} \end{aligned}$$

Because $\Lambda = c_1(N_DX^\vee)^{-1}$ and $c_1(N_DX^\vee) = -c_1(N_DX)$, we finally deduce that

$$\begin{aligned} (cl({}^t\overline{E})_* \circ cl({}^t\overline{E})_*)(\alpha) &= -n\ i_*(cl({}^tE)_*(i^*\alpha))) \\ &= -n\ cl({}^t\overline{E})_*(\alpha) \end{aligned}$$

Hence the correspondence $-\frac{1}{n}\, cl({}^t\overline{E}) \in H^6(X \times X)$ is a homological projector. □

**Remark 2.4.2.** The statement of Theorem 2.4.1 still holds when the dual of the normal bundle to $D$ in $X$ is nef and big, indeed $c_1(N_DX^\vee)$ defines a semismall map from $D$ to a projective space and it satisfies the Hard Lefschetz Theorem (see [7]).

The projection from $H^*(X)$ to $H^3(D)$ is induced by an algebraic cycle on $X \times X$:

**Theorem 2.4.2.** *There exists an algebraic cycle $\gamma_3 \in A_3(X \times X)$ whose associated cohomological class defines a motive $(X, cl(\gamma_3))$ and the cohomology of such a motive is as follows*

$$H^*((X, cl(\gamma_3))) = H^3((X, cl(\gamma_3))) \cong H^3(D) \tag{2.12}$$

*Proof.* Let us define

$$\gamma_3 := -\frac{1}{n}\, {}^t\overline{E}$$

It is a cycle in $A_3(X \times X, \mathbb{Q})$ and from Theorem 2.4.1 it follows that its cohomological class in $H^6(X \times X)$ is a projector

$$cl(\gamma_3) \circ cl(\gamma_3) = cl(\gamma_3)$$

The cohomology groups of the associated motive $(X, cl(\gamma_3))$ are by definition the images of the induced homomorphisms $cl(\gamma_3)_*$:

$$H^i((X, cl(\gamma_3))) = cl(\gamma_3)_* H^i(X)$$

Let $\alpha \in H^i(X)$ be a cohomology class on $X$, for $i = 0, \ldots, 6$. Then, as observed in the proof of Theorem 2.4.1,

$$\begin{aligned} cl(\gamma_3)_*(\alpha) &= -\frac{1}{n}\, cl({}^t\overline{E})_*(\alpha) \\ &= -\frac{1}{n}\, p_{(2X)*}((i \times i)_*(cl({}^tE) \cdot (i^*\alpha \times D))) \\ &= -\frac{1}{n}\, i_*(cl({}^tE)_*(i^*\alpha)) \end{aligned}$$

and this expression vanishes if $\alpha \in H^i(X)$ with $i \neq 3$. Therefore

$$H^i((X, cl(\gamma_3))) = 0 \tag{2.13}$$

if $i \neq 3$. From Proposition 2.4.2

$$cl({}^tE)_* : H^3(D) \longrightarrow H^1(D)$$

is an isomorphism. Because the restriction $H^3(X) \xrightarrow{i^*} H^3(D)$ is surjective it follows that

$$Im(cl({}^tE)_* \circ i^*) = H^1(D)$$

The surjectivity of $i^*$ implies the injectivity of $i_*$

$$H^1(D) \xrightarrow{i_*} H^3(X)$$

by duality. Therefore by identifying $H^1(D)$ and $H^3(D)$ in $H^3(X)$ by means of Poincaré duality we infer that

$$Im\ cl(\gamma_3)_* = Im(i_* \circ cl({}^tE)_* \circ i^*) \cong H^3(D) \tag{2.14}$$

We then obtain the cohomology of $(X, cl(\gamma_3))$

$$\begin{aligned} H^*((X, cl(\gamma_3))) &= cl(\gamma_3)_* H^*(X) \\ &= cl(\gamma_3)_* H^3(X) && \text{(by (2.13))} \\ &\cong H^3(D) && \text{(by (2.14))} \end{aligned}$$

□

We now deduce that the intersection cohomology group $IH^3(Y)$ is a direct summand of the cohomology of a homological motive and moreover, it is the cohomology of a homological motive:

**Proposition 2.4.3.** *The intersection cohomology group $IH^3(Y)$ is a direct summand of the cohomology of the homological motive $(X, cl(\Delta)_* - cl(\gamma_3)_*)$ and it is the cohomology of the homological motive $(X, \pi_{3*} - cl(\gamma_3)_*)$ where $\pi_3 \in H^3(X) \otimes H^3(X)$ is the $(3,3)$-Künneth component of $cl(\Delta) \in H^6(X \times X)$*

$$IH^3(Y) = H^*(X, \pi_{3*} - cl(\gamma_3)_*)$$

*Proof.* The cohomology group $H^3(X)$ decomposes as $H^3(D) \oplus IH^3(Y)$ (see Theorem 2.1.4). From Theorem 2.4.2 the map $cl(\gamma_3)_*$ is the projection from $H^*(X)$ to $H^3(D)$ therefore $IH^3(Y)$ is a direct summand of $Ker(cl(\gamma_3)_*)= Im(cl(\Delta)_* - cl(\gamma_3)_*)$, which is the same as $H^*(X, cl(\Delta)_* - cl(\gamma_3)_*)$.

The homological class $\pi_3$ is the $(3,3)$-Künneth component of $cl(\Delta) \in H^6(X \times X)$ and it is the projection from $H^*(X)$ to $H^3(X)$. It follows that

$$\pi_{3*} - cl(\gamma_3)_* : H^*(X) \to IH^3(Y)$$

is the projection on $IH^3(Y)$ and $\pi_{3*} - cl(\gamma_3)_*$ is a homological projector. Consequently

$$IH^3(Y) = Ker(cl(\Delta)_* - (\pi_{3*} - cl(\gamma_3)_*))$$

or equivalently

$$\begin{aligned} H^*((X, \pi_{3*} - cl(\gamma_3)_*)) &= (\pi_{3*} - cl(\gamma_3)_*) H^*(X) \\ &= (\pi_{3*} - cl(\gamma_3)_*) H^3(X) \\ &= IH^3(Y) \end{aligned}$$

□

From the results of this chapter we deduce the following statement:

**Theorem 2.4.3.** *Let $Y$ be a projective complex variety of dimension three with an isolated singular point $y$. If $f : X \to Y$ is a resolution of the singularity of $Y$ and if $D = f^{-1}(y)$ is a smooth exceptional divisor such that $N_D X^\vee$ is ample, then there exists a homological motive $(X, \gamma)$ whose cohomology is $IH^*(Y)$.*

*Proof.* Recall that $cl(\Delta) = \sum_{i=0}^{6} \pi_i$, with $\pi_i \in H^{6-i}(X) \otimes H^i(X)$, is the Künneth decomposition of the class of the diagonal $cl(\Delta) \in H^6(X \times X)$.

From what previously established, the homological cycles $cl(\gamma_i)$, for $i = 2, 3, 4$, are the projections from $H^*(X)$ to $H_4(D)$, $H^3(D)$ and $H^4(D)$ respectively and they turn out to be mutually orthogonal. Therefore they give decompositions of the respective $\pi_i = (\pi_i - cl(\gamma_i)) + cl(\gamma_i)$. By Propositions 2.2.3, 2.2.6, 2.3.2, 2.3.3, and 2.4.3, if $\gamma := \pi_0 + \pi_1 + (\pi_2 - cl(\gamma_2)) + (\pi_3 - cl(\gamma_3)) + (\pi_4 - cl(\gamma_4)) + \pi_5 + \pi_6 = cl(\Delta) - cl(\gamma_2 + \gamma_3 + \gamma_4)$ is such a homological projector, then $IH^*(Y) = H^*(X, \gamma)$. $\square$

**Remark 2.4.3.** In particular, the intersection motive of $Y$ is a direct summand of the homological motive $(X, cl(\Delta))$ of the desingularization $X$.

**Remark 2.4.4.** If in Theorem 2.4.3 we have $H^3(D) = 0$, then we can drop the assumption on the conormal $N_D X^\vee$. Moreover in this case we can state that there exists a Chow motive $(X, \gamma)$ whose cohomology is $IH^*(Y)$, where $\gamma = \Delta - \gamma_2 - \gamma_4 \in A_3(X \times X)$ (compare with the results of section 2.3).

## 2.5 An application to elliptic modular threefolds

We now give an application of the results of this chapter to the case of elliptic modular threefolds.

We review the geometry of elliptic modular threefolds with level-$N$ structure following the exposition of [14].

Fix an integer $N \geq 3$ and let $M := M_N$ be the modular curve parametrizing elliptic curves with level-$N$ structure. The analytic space $M^{an}(\mathbb{C})$ associated to $M$ is isomorphic to the quotient $\Gamma(N)\backslash\mathfrak{H}$, where $\mathfrak{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is the Poincaré upper half plane and $\Gamma(N) \subset SL_2(\mathbb{Z})$ is the subgroup of matrices congruent to the identity modulo $N$.

A smooth completion of $M$, $j : M \hookrightarrow \overline{M}$, is obtained by adjoining a finite set of cusps $M^\infty := \overline{M} \setminus M$ which parametrize generalized elliptic curves.

Since $N \geq 3$ there exists a universal elliptic curve with level $N$ structure $\phi : E \to M$. Then the universal generalized elliptic curve with level $N$ structure $\overline{\phi} : \overline{E} \to \overline{M}$ is the canonical minimal smooth completion of $\phi : E \to M$.

For $c \in M^\infty$, the fibers $\overline{E}_c := \overline{\phi}^{\,-1}(c) \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{P}^1$ are standard Néron $N$-gons and the points over $c$ in $\overline{E} \times_{\overline{M}} \overline{E}$ which are a product of two double points of $\overline{E}_c$ are rational double points.

The fiber product $\overline{E} \times_{\overline{M}} \overline{E}$ is not smooth at these points. Let

$$f : \widetilde{E} \to \overline{E} \times_{\overline{M}} \overline{E}$$

be the blow-up of $\overline{E} \times_{\overline{M}} \overline{E}$ along that set of points. Then $\widetilde{E}$ is a non-singular threefold, called an elliptic modular threefold. The $N^2 \cdot \#(M^\infty)$ components of the exceptional divisor are quadric surfaces isomorphic to $V(xy - zw) \subset \mathbb{P}^3$.

We index with $i$ the singular points of $\overline{E} \times_{\overline{M}} \overline{E}$ and for each such a point we indicate with $\gamma_2^{(i)}$ and $\gamma_4^{(i)}$ the corresponding Chow projectors that we have defined in section 2.3. Then $\sum_i (\gamma_2^{(i)} + \gamma_4^{(i)})$ is a Chow projector, indeed cycles which can be supported over distinct points are orthogonal as they are disjoint. Moreover, each component $D = D^{(i)}$ of the exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore we have $H^3(D) = 0$.

We apply Theorems 2.3.1 and 2.3.2 to the resolution $f : \widetilde{E} \to \overline{E} \times_{\overline{M}} \overline{E}$ (see also Remark 2.4.4) and we finally obtain the following

**Theorem 2.5.1.** *The intersection cohomology of the singular fiber product $\overline{E} \times_{\overline{M}} \overline{E}$ is isomorphic to the cohomology of the Chow motive $(\widetilde{E}, \Delta - \sum_i (\gamma_2^{(i)} + \gamma_4^{(i)}))$:*

$$H^*(\widetilde{E}, \Delta - \sum_i (\gamma_2^{(i)} + \gamma_4^{(i)})) \cong IH^*(\overline{E} \times_{\overline{M}} \overline{E})$$

*where $\Delta \in A_3(\widetilde{E} \times \widetilde{E})$ is the rational class of the diagonal of $\widetilde{E} \times \widetilde{E}$, the sum is taken over all the singular points of $\overline{E} \times_{\overline{M}} \overline{E}$ and, for each of such points, $\gamma_2^{(i)}$ and $\gamma_4^{(i)} \in A_3(\widetilde{E} \times \widetilde{E})$ are the Chow projectors that we have defined in section 2.3.*

**Remark 2.5.1.** In the case of elliptic modular threefolds, it is interesting to compare our results with those obtained in section 4.2 of [14]. In that paper, Gordon and Murre give a decomposition of the diagonal $\Delta \in A_3(\widetilde{E} \times \widetilde{E})$ as a sum of mutually orthogonal Chow projectors which lift the Künneth components of the diagonal modulo homological equivalence. To this aim, they construct a projector $\widetilde{\pi}_\infty \in A_3(\widetilde{E} \times \widetilde{E})$ which is the sum over all $c \in M^\infty$ of two orthogonal projectors $\widetilde{\pi}_c^{(2)}$ and $\widetilde{\pi}_c^{(4)}$ with $\widetilde{\pi}_c^{(4)} = {}^t\widetilde{\pi}_c^{(2)}$. By denoting with $\Theta_c(\boldsymbol{m})$ the components of the fiber $\widetilde{E}_c := f^{-1}(\overline{E}_c \times_{\{c\}} \overline{E}_c)$, the projector $\widetilde{\pi}_c^{(2)}$ turns out to be of the following type

$$\widetilde{\pi}_c^{(2)} = \sum_{\boldsymbol{m}} Z_c(\boldsymbol{m}) \times_{\{c\}} \Theta_c(\boldsymbol{m})$$

where $Z_c(\boldsymbol{m})$ are appropriate algebraic 1-cycles on $\widetilde{E}_c$.

Among the surfaces $\Theta_c(\boldsymbol{m})$ there are the components of the exceptional divisor corresponding to the singular points over $c$ and which in our notation are indicated with $D$. Therefore $\widetilde{\pi}_c^{(2)}$ (resp. $\widetilde{\pi}_c^{(4)}$) contains terms which are analogous to $\gamma_2^{(i)}$ (resp. $\gamma_4^{(i)}$)

$$\gamma_2^{(i)} = \deg(\eta \cdot D \cdot D)\eta \cdot D \times D$$

and which analogously determine direct summands of $H^2(\widetilde{E})$ (resp. $H^4(\widetilde{E})$).

## 2.6 On the intersection motive of a singular threefold with a divisor mapped on a curve

Let now $Y$ be a projective complex variety of dimension 3, singular in an isolated point $y$ and along a non-singular curve $C$. Let us consider $f : X \to Y$ a resolution of $Y$ and suppose $f^{-1}(y) = D$ is a smooth surface, $\dim f^{-1}(c) = 1$ for all $c \in C$ and $f_{|f^{-1}(C)} : f^{-1}(C) \to C$ is locally topologically trivial, i.e. the fibers are topologically isomorphic.

Also in this situation the restriction of $f$ to $X \setminus D$ is semismall, i.e. the set of points in $Y \setminus \{y\}$ whose fiber has dimension $i$ has codimension at least $2i$.

In §1 of [2], Borho and MacPherson first study the Decomposition Theorem for proper semismall algebraic maps. De Cataldo and Migliorini in [6], [7] and [8] widely investigate proper semismall maps and establish many results regarding such maps. In the sequel we refer to these last papers for our observations.

In this case the stratification for $f$ is given by $Y = S_3 \amalg S_1 \amalg S_0$, where $S_3 = Y \setminus \{C, y\}$, $S_1 = C$ and $S_0 = \{y\}$.

The Decomposition Theorem together with the Semisimplicity Theorem applied to $f_{|X \setminus D}$ provides the following splitting

$$Rf_{|*}\mathbb{Q}_X[3] \cong {}^p\mathcal{H}^0(Rf_{|*}\mathbb{Q}_X[3]) \cong IC_Y(L_{Y_3}) \oplus IC_{\overline{C}}(L_C) \tag{2.15}$$

By collecting these results with those in (2.2) we obtain the topological decomposition relative to $f$

$$Rf_*\mathbb{Q}_X[3] \cong H_4(D,\mathbb{Q})_y[1] \oplus IC_Y(L_{Y_3}) \oplus IC_{\overline{C}}(L_C) \oplus H^3(D,\mathbb{Q})_y \oplus H^4(D,\mathbb{Q})_y[-1]$$

where $H^*(D,\mathbb{Q})_y$ indicates the skyscraper sheaf on $Y$ supported on $y$ with stalk $H^*(D,\mathbb{Q})$.

The decomposition in (2.15) implies a decomposition of the cohomology groups of $X$ through intersection cohomology groups of $Y$ and $\overline{C}$.

In the case of semismall resolutions $f : X \to Y$, de Cataldo and Migliorini [6] construct a correspondence $\overline{\Gamma} = \amalg_l \overline{\Gamma_l}$ in a certain product space $\amalg_l \overline{Z_l} \times X$ which takes account only of the relevant strata (i.e. the strata with codimension equals to twice the dimension of the fibers) and such a correspondence determines isomorphisms of Chow groups

$$\overline{\Gamma}_* : \bigoplus_l A_*(\overline{Z_l}) \to A_*(X)$$

with $A_*(\overline{Z_l}) \to A_{*+t_l}(X)$ and isomorphisms of mixed Hodge structures

$$\overline{\Gamma}_* : \bigoplus_l H^*(\overline{Z_l})(t_l) \to H^*(X) \tag{2.16}$$

where $t_l = \frac{1}{2}(\dim X - \dim Y_l) = \dim f^{-1}(y_l)$, for $y_l \in Y_l$ relevant stratum and $H(t)$ is the Tate twist of $H$ (it decreases the weight of $H$ by $2t$) (see Theorem 4.0.4 of [6]).

For the moment we postpone the construction of the correspondence $\overline{\Gamma}$ and of the varieties $\overline{Z_l}$.

In analogy to the statement of Theorem 2.1.4, it holds the following fact (see [8]):

**Theorem 2.6.1.** *Let $f : X \to Y$ be a resolution of the singularities $y$ and $C$ of $Y$ as above. Then the intersection cohomology groups of $Y$ are direct summands of the cohomology groups of $X$ and the following cohomological decompositions hold*

$$\begin{aligned}
&H^0(X) \cong IH^0(Y)\\
&H^1(X) \cong IH^1(Y)\\
&H^2(X) \cong H_4(D) \oplus H^0(\overline{Z}) \oplus IH^2(Y)\\
&H^3(X) \cong H^3(D) \oplus H^1(\overline{Z}) \oplus IH^3(Y)\\
&H^4(X) \cong H^4(D) \oplus H^2(\overline{Z}) \oplus IH^4(Y)\\
&H^5(X) \cong IH^5(Y)\\
&H^6(X) \cong IH^6(Y)
\end{aligned} \tag{2.17}$$

*where $\overline{Z}$ is the projective compactification of the étale covering of $C$ associated with $f$.*

*Sketch of the proof.* Recall that the stratification on $Y = Y_3 \amalg Y_1 \amalg Y_0$, where $Y_0 = \{y\}$, $Y_1 = C$ and $Y_3$ is the open dense stratum, consists of only relevant strata when we restrict $f$ to $X \setminus D$.

As already observed such a restriction is semismall and in this case isomorphisms in (2.16) give a decomposition of rational cohomology of $X$ in terms of the intersection cohomology of $Y$ and of cohomology groups of $\overline{Z}$. To be more precise, if $f : X \to Y$ were the resolution of $Y$ in the case of $Y$ singular only along $C$, then we would get the decomposition of $H^*(X)$ as it is in (2.17) except for the presence of the cohomology groups of $D$. Let us note that the observations of Remark 2.1.2 still hold true. Therefore, by taking into account the decomposition theorem for $f$, we obtain the required cohomological decomposition (2.17). □

Starting from the above result, we deduce the following

**Theorem 2.6.2.** *Let $f : X \to Y$ be a resolution of $Y$ singular along $y$ and a non-singular curve $C$ such that $f^{-1}(y) = D$ is a smooth surface, $\dim f^{-1}(c) = 1$ for all $c \in C$ and $f_{|f^{-1}(C)}$ is locally topologically trivial. Let us suppose that $N_D X^\vee$ is an ample line bundle. Then there exists a homological projector $\tilde{\gamma}$ in $H^*(X \times X)$ such that*

$$IH^*(Y) \cong Im(\tilde{\gamma}_*)$$

*where $\tilde{\gamma}_* : H^*(X) \to H^*(X)$ is the homomorphism induced by $\tilde{\gamma}$.*

*Proof.* In [6] it is defined a correspondence ${}^t\overline{\Gamma'} = {}^t\overline{\Gamma'_1} \amalg {}^t\overline{\Gamma'_3}$ in $Z_*(X \times (\overline{Z_1} \amalg \overline{Z_3}))$ which gives the inverse isomorphism to (2.16).

Since the construction of such correspondences is local we can state that also in our case ${}^t\overline{\Gamma'_1}_*$ is the projection from $H^*(X)$ to $H^{*-2}(\overline{Z_1})$, where $\overline{Z_1}$ is the projective compactification of the étale covering of $C$ associated with $f$. Let $P_1$ be the projector given by $\overline{\Gamma_1} \circ {}^t\overline{\Gamma'_1}$ in $A_3(X \times X)$. It projects $H^*(X)$ to $H^*(\overline{Z_1}) \subset H^*(X)$. It follows that

$$IH^*(Y) = Im(\tilde{\gamma}_*)$$

where $\tilde{\gamma} := cl(\Delta) - cl(\gamma_2 + \gamma_3 + \gamma_4) - cl(P_1)$, and that $IH^*(Y)$ is the cohomology of a pure homological motive, $(X, \tilde{\gamma})$. □

In order to see the local nature of the correspondences in the semismall case, let us now briefly recall the construction of the correspondences $\overline{\Gamma}$ and ${}^t\overline{\Gamma'}$ that de Cataldo and Migliorini do in §4 of [6].

Let $f : X \to Y$ be a semismall resolution. Let $S_l$ be a relevant stratum of the decomposition of $Y$, $s \in S_l$, $E_l$ the set of maximal dimensional irreducible components of $f^{-1}(s)$. The group $G_l := \pi_1(S_l, s)$ acts of $E_l$ by permuting the components. Denote by $\nu_l : Z_l \to S_l$ the étale covering corresponding to $E_l$. It is possible to extend $\nu_l$ to a small map $\overline{\nu_l} : \overline{Z_l} \to \overline{S_l} \subset Y$.

The correspondence $\overline{\Gamma}$ will be supported on $\amalg_l \overline{Z_l} \times_Y X$. Each representative $o_i$, $i = 1, \ldots, r$, if the $G_l$- orbits of $E_l$ corresponds to a fixed point $s_i$ in the connected component $Z_{l,i}$ of $Z_l$. Let $G_i$ be the stabilizer of $o_i$. Then the $G_i$-orbit $\{o_i\}$ determines an irreducible component $\Gamma_{l,i}$ of $Z_{l,i} \times_Y X$. In this way we obtain $r$ irreducible varieties $\Gamma_{i,l}$ in $Z_{i,l} \times_Y X$. Define $\overline{\Gamma} := \amalg_l \overline{\Gamma_l}$ where $\overline{\Gamma_l} = \amalg_i \overline{\Gamma_{i,l}}$ is the union of the closure of the correspondences in $\overline{Z_{i,l}} \times_Y X$.

As regards the inverse correspondence ${}^t\overline{\Gamma'}$, it is constructed analogously as a union of the transpose of $\overline{\Gamma'_{i,l}}$ where $\Gamma'_{i,l}$ are defined as follows. Let $\Lambda^l = (\Lambda^l_{i,j})_{i,j \in E_l}$ be the inverse of the intersection matrix associated with the pair $(S_l, s)$. Since the intersection numbers are monodromy invariant, $\Lambda^l_{ig,jg} = \Lambda^l_{i,j}$, for all $g \in G_l$. If $I = \{f_{i_1}, \ldots, f_{i_t}\}$ is a $G_i$-orbit of components of $f^{-1}(s)$ then it corresponds to an irreducible component of $Z_{l,i} \times_Y X$. It follows that for each $i \in E_l$, $\sum_j \Lambda^l_{i,j} f_j$ is a rational linear combination of $G_i$-orbits in the set of maximal dimensional components of $f^{-1}(s)$ and it determines the correspondence $\Gamma'_{l,i}$.

# Chapter 3

# The geometry of $X[n]$

In this chapter we prove some new results on the Fulton-MacPherson compactification $X[n]$.

In several cases we will make use of the following general fact:

**Proposition 3.0.1.** *If $X, Y$, and $Z$ are irreducible algebraic schemes over a field $k$ and $X \subset Y$ is a closed imbedding then*

$$Bl_{X\times Z}(Y \times Z) \cong Bl_X Y \times Z$$

*Proof.* We can apply B.6.9. of the appendix of [12] to the first projection $p_Y : Y \times Z \to Z$ and get that

$$Bl_{X\times Z}(Y \times Z) \subset Bl_X Y \times_Y (Y \times Z)$$

is a closed embedding. Since the two spaces have the same dimension equals to $\dim Y + \dim Z$ and since $Bl_X Y \times_Y (Y \times Z)$ is irreducible, it follows that $Bl_{X\times Z}(Y \times Z) \cong Bl_X Y \times_Y (Y \times Z)$ (see for instance exercise 1.10-d at page 8 of [20]). Finally

$$Bl_X Y \times_Y (Y \times Z) \cong Bl_X Y \times Z$$

because the map

$$Bl_X Y \times_Y (Y \times Z) \to Bl_X Y \times Z : (y_1, y_2, z) \mapsto (y_1, z)$$

is an isomorphism whose inverse map sends $(y, z)$ to $(y, \pi(y), z)$, where $\pi : Bl_X Y \to Y$ is the projection, indeed $\pi(y_1) = y_2$ by the definition of a fiber product. $\square$

## 3.1 The Chow groups of $X[2], X[3]$, and $X[4]$

In this section we give explicit formulations for the Chow groups of the first three spaces for which we are able to provide a direct calculation.

**Theorem 3.1.1.** *The Chow groups of $X[2]$ are expressed by the following formula*

$$A_k(X[2]) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(X) \oplus A_k(X^2) \tag{3.1}$$

*Proof.* The case of $n=2$ is quite easy, since $X[2]$ is the blow-up of $X^2$ along the diagonal $\Delta = \Delta_{1,2}$.

The regular embedding $X \cong \Delta \hookrightarrow X \times X$ is of $\operatorname{codim}(X, X^2) = m$ and therefore by formula (1.2) it follows that

$$A_k(X[2]) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(\Delta_{1,2}) \oplus A_k(X^2) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(X) \oplus A_k(X^2)$$

□

**Theorem 3.1.2.** *The Chow groups of $X[3]$ are expressed by the following formula*

$$A_k(X[3]) \cong 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{2m-1} A_{k-i}(X) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3) \tag{3.2}$$

*Proof.* We fist recall the Fulton-MacPherson construction of $X[3]$ and then find the Chow groups in several steps. Three blow-ups are required:

$$X[3] = Y_2^{(3)} \longrightarrow Y_1^{(3)} \longrightarrow Y_0^{(3)} = X[2] \times X$$

$$X[2] \times X \longrightarrow X^2 \times X = X^3$$

The map $X[3] = Y_2^{(3)} \longrightarrow Y_1^{(3)}$ is the blow-up along $Y_1^{(3)}\{1,3\}$ and $Y_1^{(3)}\{2,3\}$. The map $Y_1^{(3)} \longrightarrow Y_0^{(3)} = X[2] \times X$ is the blow-up along $Y_0^{(3)}\{1,2,3\}$. Finally, the map $X[2] \times X \longrightarrow X^2 \times X$ is the blow-up along $\Delta_{1,2} \times X \cong X^2$.

*First step.* We apply formula (1.2) to the map $X[2] \times X \longrightarrow X^2 \times X$ and since $\operatorname{codim}(\Delta_{1,2} \times X, X^3) = \operatorname{codim}(X, X^2) = m$ it follows that

$$A_k(X[2] \times X) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(\Delta_{1,2} \times X) \oplus A_k(X^3) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3) \tag{3.3}$$

*Second step.* To find $A_k(Y_1)$ we recall the definition of $Y_0\{1,2,3\}$. It is the image of the divisor $D(1,2) \subset X[2]$ in $X[2] \times X$ under the composed embedding

$$D(1,2) \hookrightarrow D(1,2) \times X \hookrightarrow X[2] \times X$$

where the first one is the regular embedding of codimension $m$ associated with the map $D(1,2) \to \Delta_{1,2} \cong X \subset X^2$ whereas the second one is the inclusion of the exceptional divisor

$$D(1,2) \cong \mathbb{P}(N_X X^2) \cong \mathbb{P}(TX)$$

in $X[2] = Bl_X X^2$ and therefore is of codimension 1. It follows that $Y_0\{1,2,3\} \cong D(1,2)$ and $\mathrm{codim}(Y_0\{1,2,3\}, X[2] \times X) = m+1$.

We apply once again formula (1.2) and obtain

$$A_k(Y_1) \cong \bigoplus_{i=1}^{m} A_{k-i}(Y_0\{1,2,3\}) \oplus A_k(X[2] \times X) \cong \bigoplus_{i=1}^{m} A_{k-i}(D(1,2)) \oplus A_k(X[2] \times X)$$

Since $D(1,2) \cong \mathbb{P}(TX)$ and the tangent bundle is a vector bundle on $X$ of rank $m$, by formula (1.1) we deduce that

$$A_h(Y_0\{1,2,3\}) \cong A_h(D(1,2)) \cong \bigoplus_{i=0}^{m-1} A_{h-i}(X)$$

and together with formula (3.3) it follows that

$$A_k(Y_1) \cong \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3) \tag{3.4}$$

*Third step.* This is the last step since we are going to find $A_k(Y_2)$, i.e. $A_k(X[3])$. $X[3]$ is the blow-up of $Y_1$ along the subvarieties $Y_1\{1,3\}$ and $Y_1\{2,3\}$ which are respectively the strict transforms of $Y_0\{1,3\}$ and $Y_0\{2,3\}$ in the blow-up $Y_1 \longrightarrow X[2] \times X$. We notice that we can blow-up $Y_1\{1,3\}$ and $Y_1\{2,3\}$ at the same time because they are disjoint, indeed $Y_0\{1,3\}$ and $Y_0\{2,3\}$ intersect only along $Y_0\{1,2,3\}$.

By definition, $Y_0\{1,3\}$ is the image of $X[2]$ under the embedding $X[2] \hookrightarrow X[2] \times X$ given by the first projection $X[2] \to X^2 \to X$, similarly for $Y_0\{2,3\}$. Therefore $Y_1\{1,3\} \cong Y_0\{1,3\} \cong Y_1\{2,3\} \cong Y_0\{2,3\} \cong X[2]$. In addition $\mathrm{codim}(Y_1\{1,3\}, Y_1) = \mathrm{codim}(Y_1\{2,3\}, Y_1) = m$ and by formula (1.2) it follows that

$$A_k(X[3]) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(Y_1\{1,3\}) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(Y_1\{2,3\}) \oplus A_k(Y_1) \cong 2 \bigoplus_{i=1}^{m-1} A_{k-i}(X[2]) \oplus A_k(Y_1)$$

By means of (3.1) and (3.4) we infer that

$$
\begin{aligned}
&A_k(X[3])\\
&\cong 2\bigoplus_{i=1}^{m-1} A_{k-i}(X[2]) \oplus A_k(Y_1)\\
&\cong 2\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X) \oplus 2\bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(Y_1)\\
&\cong 2\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X) \oplus 2\bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3)\\
&= 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X) \oplus \bigoplus_{j=0}^{m-1} A_{k-j-m}(X) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3)\\
&= 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{2m-1} A_{k-i}(X) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3)
\end{aligned}
$$

□

**Theorem 3.1.3.** *The Chow groups of $X[4]$ are expressed by the following formula*

$$
\begin{aligned}
&A_k(X[4]) \cong\\
&= 12\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1}\bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus 7\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{2m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{3m-1} A_{k-i}(X)\\
&\oplus 15\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus 4\bigoplus_{i=1}^{2m-1} A_{k-i}(X^2)\\
&\oplus 6\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4)
\end{aligned}
\tag{3.5}
$$

*Proof.* We recall the Fulton-MacPherson construction of $X[4]$ and then analyze each one of the six blow-ups:

$$
\begin{array}{llllllll}
X[4] = Y_3^{(4)} & \longrightarrow & Y_2^{(4)} & \longrightarrow & Y_1^{(4)} & \longrightarrow & Y_0^{(4)} = X[3]\times X\\
X[3]\times X & \longrightarrow & Y_1^{(3)}\times X & \longrightarrow & Y_0^{(3)}\times X = X[2]\times X^2\\
X[2]\times X^2 & \longrightarrow & X^2\times X^2 = X^4
\end{array}
$$

The map $X[4] = Y_3^{(4)} \longrightarrow Y_2^{(4)}$ is the blow-up of $Y_2^{(4)}$ along the subvarieties $Y_2^{(4)}\{1,4\}, Y_2^{(4)}\{2,4\}$, and $Y_2^{(4)}\{3,4\}$. The map $Y_2^{(4)} \longrightarrow Y_1^{(4)}$ is the blow-up of $Y_1^{(4)}$ along $Y_1^{(4)}\{1,2,4\}, Y_1^{(4)}\{1,3,4\}$, and $Y_1^{(4)}\{2,3,4\}$. The map $Y_1^{(4)} \longrightarrow Y_0^{(4)} = X[3] \times X$ is the blow-up of $Y_0^{(4)}$ along $Y_0^{(4)}\{1,2,3,4\}$.

The second line is made of the blow-ups one needs to construct $X[3]$ multiplied by $X$, whereas the third line consists of the blow-up which determines $X[2]$ multiplied by $X^2$.

We deduce the Chow groups of $X[4]$ after the study of the intermediate spaces which we carry out in six steps.

*First step.* The map $X[2]\times X^2 \longrightarrow X^2\times X^2 = X^4$ is the blow-up along $\Delta_{1,2}\times X^2 \cong X^3$. Since $\operatorname{codim}(\Delta_{1,2} \times X^2, X^4) = m$ it follows from formula (1.2) that

$$A_k(X[2] \times X^2) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \tag{3.6}$$

*Second step.* We are now going to find the groups $A_k(Y_1^{(3)} \times X)$. This time we need to blow-up $Y_0^{(3)}\{1,2,3\} \times X$ inside $Y_0^{(3)} \times X = X[2] \times X^2$ and since $\operatorname{codim}(Y_0^{(3)}\{1,2,3\} \times X, X[2] \times X^2) = \operatorname{codim}(Y_0^{(3)}\{1,2,3\}, X[2] \times X) = m+1$, by formula (1.2) we have

$$A_k(Y_1^{(3)} \times X) \cong \bigoplus_{i=1}^{m} A_{k-i}(Y_0^{(3)}\{1,2,3\} \times X) \oplus A_k(X[2] \times X^2)$$

As observed in the proof of Theorem (3.1.2) at the second step, the variety $Y_0^{(3)}\{1,2,3\}$ is isomorphic to $D(1,2) \cong \mathbb{P}(N_X X^2)$. Since $X[2] \times X = (Bl_X X^2) \times X \cong Bl_{X^2} X^3$, it follows that $Y_0^{(3)}\{1,2,3\} \times X \cong \mathbb{P}(N_{X^2} X^3)$ and by formula (1.1) we deduce the Chow groups of $Y_0^{(3)}\{1,2,3\} \times X$:

$$A_k(Y_0^{(3)}\{1,2,3\} \times X) \cong \bigoplus_{i=0}^{m-1} A_{k-i}(X^2)$$

By formula (3.6) and the expression above we find

$$A_k(Y_1^{(3)} \times X) \cong \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \tag{3.7}$$

*Third step.* As seen in the proof of Theorem (3.1.2), the subvarieties $Y_1^{(3)}\{1,3\}$ and $Y_1^{(3)}\{2,3\}$ are both isomorphic to $X[2]$, in addition $\operatorname{codim}(Y_1^{(3)}\{1,3\} \times X, Y_1^{(3)} \times X) = \operatorname{codim}(Y_1^{(3)}\{1,3\}, Y_1^{(3)}) = m$ therefore, by formula (1.2),

$$\begin{aligned} A_k(X[3] \times X) &\cong \bigoplus_{i=1}^{m-1} A_{k-i}(Y_1^{(3)}\{1,3\} \times X) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(Y_1^{(3)}\{2,3\} \times X) \oplus A_k(Y_1^{(3)} \times X) \\ &\cong 2\bigoplus_{i=1}^{m-1} A_{k-i}(X[2] \times X) \oplus A_k(Y_1^{(3)} \times X) \end{aligned}$$

Regarding $X[2] \times X$ as $Bl_{X^2}X^3$ we can write

$$A_k(X[2] \times X) \cong \bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus A_k(X^3)$$

By formula (3.7) and the expression above it follows then that

$$\begin{aligned}
&A_k(X[3] \times X) \\
&\cong 2\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus 2\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \\
&= 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{j=0}^{m-1} A_{k-m-j}(X^2) \oplus \bigoplus_{i=1}^{m-1} A_{k-i}(X^2) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \\
&= 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4)
\end{aligned} \tag{3.8}$$

*Fourth step.* $Y_0^{(4)}\{1,2,3,4\}$ is the center of the next blow-up, $Y_1^{(4)} \longrightarrow Y_0^{(4)} = X[3] \times X$. By definition this variety is the image of $D(1,2,3)$ (divisor inside $X[3]$) in $X[3] \times X$ under the composed embedding

$$D(1,2,3) \hookrightarrow D(1,2,3) \times X \hookrightarrow X[3] \times X$$

where the first one is the embedding associated with the map $D(1,2,3) \to \Delta_{1,2,3} \cong X \subset X^3$ and therefore is of codimension $m$, the second one is just the inclusion $D(1,2,3) \subset X[3]$.

The divisor $D(1,2,3) \subset X[3]$ is the exceptional divisor of $Y_1^{(3)} \to X[2] \times X$ therefore $Y_0^{(4)}\{1,2,3,4\}$ is isomorphic to a projective bundle on $\mathbb{P}(TX)$ of rank $m$

$$Y_0^{(4)}\{1,2,3,4\} \cong D(1,2,3) \cong \mathbb{P}(N_{Y_0^{(3)}\{1,2,3\}}(X[2] \times X)) \cong \mathbb{P}(N_{\mathbb{P}(TX)}(X[2] \times X))$$

Since $\operatorname{codim}(Y_0^{(4)}\{1,2,3,4\}, Y_0^{(4)} = X[3] \times X) = m+1$ and $\operatorname{codim}(Y_0^{(3)}\{1,2,3\}, Y_0^{(3)} = X[2] \times X) = m+1$, by means of formulae (1.2) and (1.1), applied first to $\mathbb{P}(N_{Y_0^{(3)}\{1,2,3\}}(X[2] \times$

$X$)) and then to $\mathbb{P}(TX)$, we infer that

$$\begin{aligned}
&A_k(Y_1^{(4)}) \\
&\quad\cong \bigoplus_{i=1}^{m} A_{k-i}(Y_0^{(4)}\{1,2,3,4\}) \oplus A_k(X[3]\times X) \\
&\quad\cong \bigoplus_{i=1}^{m} A_{k-i}(\mathbb{P}(N_{\mathbb{P}(TX)}(X[2]\times X))) \oplus A_k(X[3]\times X) \\
&\quad\cong \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m} A_{k-i-j}(\mathbb{P}(TX)) \oplus A_k(X[3]\times X) \\
&\quad\cong \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m}\bigoplus_{l=0}^{m-1} A_{k-i-j-l}(X) \oplus A_k(X[3]\times X)
\end{aligned}$$

By replacing $A_k(X[3]\times X)$ with the expression determined in formula (3.8), the calculation above shows that $A_k(Y_1^{(4)})$ is isomorphic to

$$\bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m}\bigoplus_{l=0}^{m-1} A_{k-i-j-l}(X) \oplus 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \tag{3.9}$$

*Fifth step.* Since $Y_2^{(4)}$ is the blow-up of $Y_1^{(4)}$ along the subvarieties $Y_1^{(4)}\{i_1,i_2,4\}$, for $i_1 < i_2 \in \{1,2,3\}$, and they all have codimension $m+1$, again by means of formula (1.2) it follows that

$$A_k(Y_2^{(4)}) \cong \bigoplus_{i=1}^{m} A_{k-i}(Y_1^{(4)}\{1,2,4\}) \oplus \bigoplus_{i=1}^{m} A_{k-i}(Y_1^{(4)}\{1,3,4\}) \bigoplus_{i=1}^{m} A_{k-i}(Y_1^{(4)}\{2,3,4\}) \oplus A_k(Y_1^{(4)})$$

By definition $Y_1^{(4)}\{1,2,4\}$ is the strict transform of $Y_0^{(4)}\{1,2,4\}$ in the map $Y_1^{(4)} \longrightarrow Y_0^{(4)} = X[3]\times X$ and in addition $Y_0^{(4)}\{1,2,4\}$ is the image of $D^{(3)}(1,2)$ (divisor inside $X[3]$) under the composed embedding

$$D^{(3)}(1,2) \hookrightarrow D^{(3)}(1,2)\times X \hookrightarrow X[3]\times X$$

where the first one is induced by the projection $D^{(3)}(1,2) \to \Delta \subset X^2 = X^S$, with $S = \{1,2\}$. The divisor $D^{(3)}(1,2)$ is isomorphic to $D^{(2)}(1,2)\times X$.

We then state the following isomorphisms

$$Y_1^{(4)}\{1,2,4\} \cong Y_0^{(4)}\{1,2,4\} \cong D^{(3)}(1,2) \cong D^{(2)}(1,2)\times X \cong \mathbb{P}(N_{X^2}X^3) \tag{3.10}$$

where the last isomorphism is due to the fact that $D^{(2)}(1,2)$ is the exceptional divisor of $Bl_X X^2$.

As regards $Y_1^{(4)}\{1,3,4\}$, it is the strict transform of $Y_0^{(4)}\{1,3,4\}$ in the map $Y_1^{(4)} \longrightarrow Y_0^{(4)} = X[3] \times X$. By definition $Y_0^{(4)}\{1,3,4\}$ is the image of $D^{(3)}(1,3)$ (divisor inside $X[3]$) under the embedding

$$D^{(3)}(1,3) \hookrightarrow D^{(3)}(1,3) \times X \hookrightarrow X[3] \times X$$

where the first map is induced by the projection $D^{(3)}(1,3) \to \Delta \subset X^2 = X^S$, with $S = \{1,3\}$. The divisor $D^{(3)}(1,3)$ is isomorphic to $\mathbb{P}(N_{Y_1^{(3)}\{1,3\}} Y_1^3)$ since it is the component of the exceptional divisor of the blow-up $X[3] \to Y_1^{(3)}$ with respect to $Y_1^{(3)}\{1,3\}$.

It follows then that

$$Y_1^{(4)}\{1,3,4\} \cong \mathbb{P}(N_{Y_1^{(3)}\{1,3\}} Y_1^{(3)}) \tag{3.11}$$

and by an analogous argument that

$$Y_1^{(4)}\{2,3,4\} \cong \mathbb{P}(N_{Y_1^{(3)}\{2,3\}} Y_1^{(3)}) \tag{3.12}$$

Applying once again the formula which expresses the Chow groups of a blow-up together with formulae (3.10), (3.11), and (3.12) we deduce that

$$\begin{aligned} &A_k(Y_2^{(4)}) \\ &\cong \bigoplus_{i=1}^{m} A_{k-i}(\mathbb{P}(N_{X^2}X^3)) \oplus \bigoplus_{i=1}^{m} A_{k-i}(\mathbb{P}(N_{Y_1^{(3)}\{1,3\}} Y_1^{(3)})) \oplus \bigoplus_{i=1}^{m} A_{k-i}(\mathbb{P}(N_{Y_1^{(3)}\{2,3\}} Y_1^{(3)})) \\ &\quad \oplus A_k(Y_1^{(4)}) \end{aligned}$$

By the theorem giving a formula for the Chow groups of a projective bundle, since $\operatorname{codim}(X^2, X^3) = \operatorname{codim}(Y_1^{(3)}\{1,3\}, Y_1^3) = \operatorname{codim}(Y_1^{(3)}\{2,3\}, Y_1^3) = m$ and since $Y_1^{(3)}\{1,3\} \cong Y_1^{(3)}\{2,3\} \cong X[2]$, the previous expression of $A_k(Y_2^{(4)})$ takes the form

$$A_k(Y_2^{(4)}) \cong \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X^2) \oplus 2 \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X[2]) \oplus A_k(Y_1^{(4)})$$

By formulae (3.1) and (3.9) we derive the expression

$$\begin{aligned} &\bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X^2) \oplus 2 \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} \bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus 2 \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m-1} A_{k-i-j}(X^2) \\ &\oplus \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{m} \bigoplus_{l=0}^{m-1} A_{k-i-j-l}(X) \oplus 3 \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus \bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \\ &\oplus 3 \bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \end{aligned}$$

Finally, collecting the repeated terms, we infer that

$$\begin{aligned} &A_k(Y_2^{(4)}) \\ &\cong 3\bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m-1}\bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{2m-1} A_{k-i-j}(X) \\ &\oplus 6\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus 4\bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \\ &\oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \end{aligned} \tag{3.13}$$

*Sixth step.* The last step to determine $A_k(X[4])$ consists of the analysis of the blow-up of $Y_2^{(4)}$ along the subvarieties $Y_2^{(4)}\{1,4\}$, $Y_2^{(4)}\{2,4\}$ and $Y_2^{(4)}\{3,4\}$.

They are respectively the strict transforms of $Y_1^{(4)}\{1,4\}$, $Y_1^{(4)}\{2,4\}$ and $Y_1^{(4)}\{3,4\}$ under the map $Y_2^{(4)} \to Y_1^{(4)}$, moreover, these three are the strict transforms of $Y_0^{(4)}\{1,4\}$, $Y_0^{(4)}\{2,4\}$ and $Y_0^{(4)}\{3,4\}$ under the map $Y_1^{(4)} \to Y_0^{(4)}$. Each one of the subvarieties $Y_0^{(4)}\{i,4\}$, for $i = 1,\ldots,3$, is by definition the image of $X[3]$ under the embedding $X[3] \hookrightarrow X[3] \times X$ which is induced by the projection $X[3] \to X^3 \to X^S = X$, where $S$ takes values $S = \{i\}$, for $i = 1,2,3$ respectively.

Since $\operatorname{codim}(Y_2^{(4)}\{i,4\}, Y_2^{(4)}) = m$, by formula (1.2) we find initially

$$A_k(X[4]) \cong 3\bigoplus_{i=1}^{m-1} A_{k-i}(X[3]) \oplus A_k(Y_2^{(4)})$$

By means of expressions (3.2) and (3.13) we may rewrite the preceding isomorphism as

$$\begin{aligned} &A_k(X[4]) \cong \\ &\cong 9\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1}\bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus 3\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{2m-1} A_{k-i-j}(X) \oplus 9\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \\ &\oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus 3\bigoplus_{i=1}^{m}\bigoplus_{j=0}^{m-1}\bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus \bigoplus_{i=1}^{m}\bigoplus_{j=0}^{2m-1} A_{k-i-j}(X) \\ &\oplus 6\bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus 4\bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \oplus 3\bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4) \end{aligned}$$

Collecting the terms which are repeated we finally obtain the following formula for the

Chow groups of $X[4]$

$$
\begin{aligned}
&A_k(X[4]) \cong \\
&\quad 12 \bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1}\bigoplus_{l=1}^{m-1} A_{k-i-j-l}(X) \oplus 7 \bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{2m-1} A_{k-i-j}(X) \oplus \bigoplus_{i=1}^{3m-1} A_{k-i}(X) \\
&\quad \oplus 15 \bigoplus_{i=1}^{m-1}\bigoplus_{j=1}^{m-1} A_{k-i-j}(X^2) \oplus 4 \bigoplus_{i=1}^{2m-1} A_{k-i}(X^2) \\
&\quad \oplus 6 \bigoplus_{i=1}^{m-1} A_{k-i}(X^3) \oplus A_k(X^4)
\end{aligned}
$$

□

## 3.2 On the Chow groups of $X[n]$

Our aim is now to determine the Chow groups of the exceptional divisors of $X[n]$ since we need them in calculating those of the compactification, as it will be clear in the sequel.

**Lemma 3.2.1.** *For all $n \geq 2$ and for all subsets $S = \{s_1, \ldots, s_t\} \subseteq N = \{1, \ldots, n\}$ with at least two elements, the products of the exceptional divisors of $X[n]$ with $X^q$, for $q \geq 0$, are isomorphic to the following projective bundles*

$$D^{(n)}(S) \times X^q \cong \mathbb{P}(N_{Y^{(s_t)}_{s_t-t}(S)\times X^{n-s_t+q}}(Y^{(s_t)}_{s_t-t} \times X^{n-s_t+q}))$$

*Proof.* Let us consider first the case $q = 0$.

If $S$ contains the index $n$, that is $s_t = n$, then $D^{(n)}(S)$ is an exceptional divisor resulting from the last sequence of blow-ups, that one from $X[n-1] \times X$ to $X[n]$. In this case the isomorphism is clear.

Otherwise, if $S \subseteq N \setminus \{n\}$, $D^{(n)}(S)$ is an exceptional divisor obtained in a blow-up between $X[n-1] \times X$ and $X^n$. It is therefore an exceptional divisor appearing for the first time in the determination of $X[s_t]$ and $D^{(n)}(S)$ is isomorphic to

$$D^{(s_t)}(S) \times X^{n-s_t}$$

By means of Proposition (3.0.1) it is isomorphic to

$$\mathbb{P}(N_{Y^{(s_t)}_{s_t-t}\{s_1,\ldots,s_t\}\times X^{n-s_t}}(Y^{(s_t)}_{s_t-t} \times X^{n-s_t}))$$

and when we consider the product $D^{(n)}(S) \times X^q$ we obtain exactly the isomorphism stated. □

**Theorem 3.2.1.** *For all $n \geq 2$ and for all subsets $S = \{s_1, \dots, s_t\} \subseteq N = \{1, \dots, n\}$ with at least two elements, the following isomorphisms hold $\forall k$, $\forall q \geq 0$*

$$A_k(D^{(n)}(S) \times X^q) \cong \bigoplus_{i_1=0}^{m} \cdots \bigoplus_{i_{(t-2)}=0}^{m} \bigoplus_{i_{(t-1)}=0}^{m-1} A_{k-i_1-\dots-i_{(t-2)}-i_{(t-1)}}(X[s_2-1] \times X^{n-s_2-(t-2)+q})$$

*if $|S| = t \geq 3$, otherwise*

$$A_k(D^{(n)}\{s_1, s_2\} \times X^q) \cong \bigoplus_{i=0}^{m-1} A_{k-i}(X[s_2-1] \times X^{n-s_2+q})$$

*Proof.* We proceed by induction on $n$.

When $n = 2$, there is only one exceptional divisor in $X[2]$, $D^{(2)}\{1, 2\}$, which by definition is the projective bundle

$$D^{(2)}\{1, 2\} = \mathbb{P}(N_\Delta X^2)$$

therefore

$$D^{(2)}\{1, 2\} \times X^q \cong \mathbb{P}(N_{\Delta \times X^q} X^{2+q})$$

and by formula (1.1) we deduce that for all $k$ and $q$

$$A_k(D^{(2)}\{1, 2\} \times X^q) \cong \bigoplus_{i=0}^{m-1} A_{k-i}(X^{1+q})$$

and the statement is proved since we have defined $X[1] = X$.

Let us now suppose that the theorem holds for all indexes $2 \leq p \leq n-1$, for $n \geq 3$, and show that it is true for $n$. Let us fix $q \geq 0$ and $k$. From the previous lemma, we know that

$$D^{(n)}\{s_1, \dots, s_t\} \times X^q = D^{(s_t)}\{s_1, \dots, s_t\} \times X^{n-s_t} \times X^q$$

If $s_t < n$ then, by induction, the group $A_k(D^{(n)}\{s_1, \dots, s_t\} \times X^q)$ is isomorphic to

$$\bigoplus_{i_1=0}^{m} \cdots \bigoplus_{i_{t-2}=0}^{m} \bigoplus_{i_{t-1}=0}^{m-1} A_{k-i_1-\dots-i_{(t-2)}-i_{(t-1)}}(X[s_2-1] \times X^{s_t-s_2-(t-2)+n-s_t+q})$$

if $t \geq 3$, otherwise, for $t = 2$, it is isomorphic to

$$\bigoplus_{i=0}^{m-1} A_{k-i}(X[s_2-1] \times X^{n-s_2+q})$$

When $s_t = n$ then the product $D^{(n)}\{s_1, \dots, s_t\} \times X^q$ is just the projective bundle

$$\mathbb{P}(N_{Y_{n-t}^{(n)}\{s_1,\dots,s_t\} \times X^q}(Y_{n-t}^{(n)} \times X^q))$$

Since the subvariety $Y^{(n)}_{n-t}\{s_1,\dots,s_t\}$ is isomorphic to $Y^{(n)}_0\{s_1,\dots,s_t\}$ which is $X[n-1]$ if $t=2$, whereas it is $D^{(n-1)}\{s_1,\dots,s_{(t-1)}\}$ in the case $t\geq 3$, it follows that

$$A_k(D^{(n)}\{s_1,\dots,s_t\}\times X^q)\cong\begin{cases}\bigoplus_{i=0}^{m-1}A_{k-i}(X[n-1]\times X^q) & t=2\\ \bigoplus_{i=0}^{m}A_{k-i}(D^{(n-1)}\{s_1,\dots,s_{(t-1)}\}\times X^q) & t\geq 3\end{cases}$$

By induction, the second case becomes

$$\bigoplus_{i=0}^{m}\bigoplus_{i_1=0}^{m}\cdots\bigoplus_{i_{t-3}=0}^{m}\bigoplus_{i_{t-2}=0}^{m-1}A_{k-i-i_1-\dots-i_{t-3}-i_{t-2}}(X[s_2-1]\times X^{n-1-s_2-(t-3)+q})$$

and we are done. □

**Corollary 3.2.1.** *For all $n\geq 2$ and for all subsets $S=\{s_1,\dots,s_t\}\subseteq N=\{1,\dots,n\}$ with at least two elements, the Chow groups of the exceptional divisors $D^{(n)}(S)$ of $X[n]$ are given by the following expression*

$$A_k(D^{(n)}(S))\cong\bigoplus_{i_1=0}^{m}\cdots\bigoplus_{i_{t-2}=0}^{m}\bigoplus_{i_{t-1}=0}^{m-1}A_{k-i_1-\dots-i_{(t-2)}-i_{(t-1)}}(X[s_2-1]\times X^{n-s_2-(t-2)})\quad(3.14)$$

*if $|S|\geq 3$, when $|S|=2$ they are*

$$A_k(D^{(n)}(S))\cong\bigoplus_{i=0}^{m-1}A_{k-i}(X[s_2-1]\times X^{n-s_2})\quad(3.15)$$

*for all $0\leq k\leq mn-1$.*

*Proof.* Apply Theorem (3.2.1) with $q=0$. □

**Remark 3.2.1.** From the corollary above it is clear that the exceptional divisors of $X[n]$ which correspond to subsets of labels with the same $s_2-$term and the same number of elements $|S|$ have isomorphic Chow groups.

The following two lemmas will be useful in the proof of the next theorem.

**Lemma 3.2.2.** *The direct sum of all $A_k(D^{(n)}(S))$ as $S$ varies among the subsets of $N=\{1,\dots,n\}$ with at least two elements, for a fixed $k$, can be expressed in the following manner*

$$\bigoplus_{\substack{S\subseteq N\\|S|\geq 2}}A_k(D^{(n)}(S))\cong$$

$$\bigoplus_{r=2}^{n}(r-1)\bigoplus_{i=0}^{m-1}A_{k-i}(X[r-1]\times X^{n-r})\oplus$$

$$\bigoplus_{t=3}^{n}\bigoplus_{r=2}^{n-t+2}(r-1)\binom{n-r}{t-2}\bigoplus_{i_1=0}^{m}\cdots\bigoplus_{i_{(t-2)}=0}^{m-1}\bigoplus_{i_{(t-1)}=0}^{m-1}A_{k-i_1-\dots-i_{(t-2)}-i_{(t-1)}}(X[r-1]\times X^{n-t+2-r})$$

*Proof.* Each time we fix $t = 2, \dots, n$ and $r = 2, \dots, n$, there are

$$(r-1)\binom{n-r}{t-2}$$

exceptional divisors $D^{(n)}(S)$ of $X[n]$, such that $|S| = t$ and $s_2 = r$. Indeed we can choose $s_1$ in $r-1$ ways whereas $\{s_3 < \cdots < s_t\}$ are $t-2$ indexes which can vary between $r+1$ and $n$. In particular, when $t$ is fixed, $r$ must be at most $n-t+2$.

From what we have observed in remark (3.2.1), we can collect the divisors wich correspond to sets of labels $S$ with the same values of $|S| = t$ and $s_2 = r$ therefore, by means of Corollary (3.2.1), we infer that $\bigoplus_{\substack{S\subseteq N\\ |S|\geq 2}} A_k(D^{(n)}(S))$ is isomorphic to the expression given in the statement of the lemma. $\square$

**Lemma 3.2.3.** *The intermediate varieties $Y_h^{(n)}$ obtained in the sequence of blow-ups between $X[n]$ and $X[n-1]\times X$ have Chow groups given by*

$$A_k(Y_h^{(n)}) \cong \bigoplus_{i=1}^{m} \bigoplus_{\substack{S\subseteq N'=\{1,\dots,n-1\}\\ |S|\geq 2\\ |S|\geq n-h}} A_{k-i}(D^{(n-1)}(S)) \oplus A_k(Y_0^{(n)})$$

*for all $n \geq 3$, $k$ and $h = 1, \dots, n-2$.*

*Proof.* We proceed by induction on $n$.

When $n = 3$, $h$ is 1 and by formula (1.2) we know that

$$A_k(Y_1^{(3)}) \cong \bigoplus_{i=1}^{m} A_{k-i}(D^{(2)}(1,2)) \oplus A_k(Y_0^{(3)})$$

for all $k$, indeed $Y_0^{(3)}\{1,2,3\} \cong D^{(2)}(1,2)$.

Let us now suppose the statement is true for $n-1$ and see it is satisfied for $n$. We will prove the assertion by induction on $h$. If $h = 1$ then

$$A_k(Y_1^{(n)}) \cong \bigoplus_{i=1}^{m} A_{k-i}(D^{(n-1)}(1,\dots,n-1)) \oplus A_k(Y_0^{(n)})$$

for all $k$. We now suppose the isomorphism holds for $h-1$ and show it for $h$ and therefore for all $n$, $h$ and $k$. Let us fix $k$. Then

$$A_k(Y_h^{(n)}) \cong \bigoplus_{i=1}^{m} \bigoplus_{\substack{S\subseteq N'=\{1,\dots,n-1\}\\ |S|=n-h}} A_{k-i}(D^{(n-1)}(S)) \oplus A_k(Y_{h-1}^{(n)})$$

By induction we can express $A_k(Y_{h-1}^{(n)})$ in the desired way and it follows the isomorphism of the statement. $\square$

**Theorem 3.2.2.** *For all $n \geq 2$ the Chow groups of the product $X[n] \times X^q$ are given by the following isomorphisms*

$$
\begin{aligned}
&A_k(X[n] \times X^q) \cong \\
&(n-1)\bigoplus_{i=1}^{m-1} A_{k-i}(X[n-1] \times X^q) \oplus \bigoplus_{s=2}^{n-1}(s-1)\bigoplus_{i=0}^{m-1} A_{k-i}(X[s-1] \times X^{n-1-s+q}) \\
&\oplus \bigoplus_{t=2}^{n-1}\bigoplus_{s=2}^{n-1-(t-2)}(s-1)\binom{n-1-s}{t-2}\bigoplus_{i_1=1}^{m}\bigoplus_{i_2=0}^{m}\cdots\bigoplus_{i_{t-1}=0}^{m}\bigoplus_{i_t=0}^{m-1} A_{k-\sum_{j=1}^t i_j}(X[s-1] \times X^{n-1-s-(t-2)+q}) \\
&\oplus A_k(X[n-1] \times X^{1+q})
\end{aligned}
$$

*$\forall k$ and $\forall q \geq 0$.*

*Proof.* Recall that $X[n]$ is obtained by blowing up the varieties $Y^{(n)}_{n-2}(\{j,n\})$ inside $Y^{(n)}_{n-2}$, where $j = 1, \ldots, n-1$. As already observed, these subvarieties are all isomorphic to the variety $X[n-1]$, in addition their codimension is $m$. Therefore, by formula (1.2) and Proposition (3.0.1), we deduce that

$$A_k(X[n] \times X^q) \cong (n-1)\bigoplus_{i=1}^{m-1} A_{k-i}(X[n-1] \times X^q) \oplus A_k(Y^{(n)}_{n-2} \times X^q)$$

for all $k$.

By Lemma (3.2.3) and Proposition (3.0.1) we can develop $A_k(Y^{(n)}_{n-2} \times X^q)$ obtaining

$$
\begin{aligned}
&A_k(X[n] \times X^q) \cong \\
&(n-1)\bigoplus_{i=1}^{m-1} A_{k-i}(X[n-1] \times X^q) \oplus \bigoplus_{i=1}^{m}\bigoplus_{\substack{S\subseteq N' \\ |S|\geq 2}} A_{k-i}(D^{(n-1)}(S) \times X^q) \oplus A_k(Y^{(n)}_0 \times X^q)
\end{aligned}
$$

where we have indicated with $N'$ the set $\{1, \ldots, n-1\}$.

By Lemma (3.2.2) we exploit the term

$$\bigoplus_{\substack{S\subseteq N' \\ |S|\geq 2}} A_{k-i}(D^{(n-1)}(S) \times X^q)$$

as a combination of the Chow groups of products of the type $X[p] \times X^q$ and since $Y^{(n)}_0$ is $X[n-1]$, we are done. $\square$

As a consequence of the previous theorem we deduce the Chow groups of the compactification $X[n]$ which result linear combinations of the Chow groups of the varieties of the type $X[p] \times X^q$ in a sort of inductive formula since $p \leq n-1$ and $p+q \leq n$:

**Corollary 3.2.2.** *For all $n \geq 2$ the Chow groups of $X[n]$ are given by the following expression*

$$A_k(X[n]) \cong$$

$$(n-1)\bigoplus_{i=1}^{m-1} A_{k-i}(X[n-1]) \oplus \bigoplus_{s=2}^{n-1}(s-1)\bigoplus_{i=0}^{m-1} A_{k-i}(X[s-1]\times X^{n-1-s})$$

$$\oplus \bigoplus_{t=2}^{n-1}\bigoplus_{s=2}^{n-1-(t-2)}(s-1)\binom{n-1-s}{t-2}\bigoplus_{i_1=1}^{m}\bigoplus_{i_2=0}^{m}\cdots\bigoplus_{i_{t-1}=0}^{m}\bigoplus_{i_t=0}^{m-1} A_{k-\sum_{j=1}^{t} i_j}(X[s-1]\times X^{n-1-s-(t-2)})$$

$$\oplus A_k(X[n-1]\times X)$$

*for all $0 \leq k \leq mn$.*

*Proof.* Apply Theorem (3.2.2) with $q = 0$. □

**Corollary 3.2.3.** *Each Chow group of $X[n]$ is isomorphic to a direct sum of Chow groups of the products $X^p$, with $p = 1, \ldots, n-1$.*

*In addition, the Chow groups of the exceptional divisors $D^{(n)}(S)$ of $X[n]$ are direct sums of the Chow groups of the powers $X^q$, for $q = n-t+1-(s_2-2), \ldots, n-t+1$.*

*Proof.* As for the first part of the statement, first we observe from Corollary (3.2.2) that the varieties appearing on the right of the formula are of the type $X[y]\times X^z$ with $y \leq n-1$ and $y+z \leq n$. Then, for our aim, it is sufficient to note that the group $A_k(X[n])$ depends on $A_{k-i}(X[n-1])$, for some $i$, and $A_k(X[n-1]\times X)$, as we can see in Corollary (3.2.2). In particular, by an inductive proceed, developing the second group we find $A_k(X^n)$, as the index of $X[n-1]$ decreases by one and the power of $X$ increases by one at each step (see Theorem (3.2.2)) and therefore, when we exploit the first groups, the groups $A_{k-i}(X^p)$ appear, for all $p = 1, \ldots, n-1$.

With the same argument, by means of Corollary (3.2.1) and Theorem (3.2.2), we deduce the second part of the statement for the divisors $D^{(n)}(S)$. □

**• A numeric interpretation.**

We now want to rewrite the numeric part of the formula in Theorem (3.2.2). Indeed, if the groups $A_k(X[n]\times X^q)$ are finitely generated, or equivalently if $A_k(X^p)$ are so, for all $p \leq n+q$, we indicate with $f(k,n,q)$ the dimension of $A_k(X[n]\times X^q)$.

It is useful to introduce the following notion which generalizes some tools from Chapter I of [31]:

**Definition 3.2.1.** *For all $t \geq 1$ and for all $m \geq 1$ we define $r_t(l)$ as follows*

$$r_t(l) := \#\{(i_1,\ldots,i_t)\in\{0,\ldots,m\}^t | i_1+\cdots+i_t = l\}$$

*Note that $r_t(l)$ is the coefficient of $z^l$ in the polynomial*

$$(1+z+\cdots+z^m)^t$$

*indeed $\sum r_t(l)z^l = A^t(z)$, where $A(t) = \sum_{a\in A} z^a$ and $A = \{0,\ldots,m\}$.*

**Proposition 3.2.1.** *If we indicate with $F(k,q)$ the generating function of $f(k,n,q)$ with respect to $n$*

$$F(k,q)=\sum_{n\geq 0} f(k,n,q)z^n$$

*then this function verifies the following relation*

$$\begin{aligned}
F(k,q) &= z^2\frac{d}{dz}\sum_{i=1}^{m-1}F(k-i,q)+zF(k,q+1)+\sum_{\substack{\sigma,\rho\geq 0\\ \sigma+\rho\geq 3}} z^{\sigma+3}\frac{d}{dz}\Big(\sum_{i=0}^{m-1}F(k-i,\rho+q)\Big)+\\
&\sum_{\substack{\tau\geq 1\\ \sigma,\rho\geq 0\\ \tau+\sigma+\rho\geq 3}} z^{\rho+\tau+3}\frac{d}{dz}\Big(\binom{\rho+\tau}{\tau}\sum_{l=0}^{\infty}\big(r_{\tau+2}(l)F(k-l,\rho+q)-r_{\tau+1}(l)F(k-l-m,\rho+q)+\\
&r_\tau(l)F(k-l-m,\rho+q)-r_{\tau+1}(l)F(k-l,\rho+q)\big)\Big)
\end{aligned} \tag{3.16}$$

*Proof.* From Theorem (3.2.2) we deduce that $f(k,n,q)$ is given by

$$\begin{aligned}
&f(k,n,q)=\\
&(n-1)\sum_{i=1}^{m-1}f(k-i,n-1,q)+\\
&\sum_{s=2}^{n-1}\sum_{i=0}^{m-1}(s-1)f(k-i,s-1,n-1-s+q)+\\
&\sum_{t=3}^{n-1}\sum_{s=2}^{n-t+1}(s-1)\binom{n-1-s}{t-2}\sum_{i_1=1}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{t-1}=0}^{m}\sum_{i_t=0}^{m-1}f(k-\sum_{j=1}^{t}i_j,s-1,n-s-t+1+q)+\\
&f(k,n-1,q+1)
\end{aligned} \tag{3.17}$$

The term

$$\sum_{i_1=1}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{t-1}=0}^{m}\sum_{i_t=0}^{m-1}f(k-\sum_{j=1}^{t}i_j,s-1,n-s-t+1+q)$$

can be rewritten in a more compact way. Indeed by a simple calculation we find it is equals to

$$\begin{aligned}
&\sum_{l=0}^{mt}r_t(l)f(k-l,s-1,\dots)-\sum_{l=0}^{m(t-1)}r_{t-1}(l)f(k-l-m,s-1,\dots)\\
&+\sum_{l=0}^{m(t-2)}r_{t-2}(l)f(k-l-m,s-1,\dots)-\sum_{l=0}^{m(t-1)}r_{t-1}(l)f(k-l,s-1,\dots)
\end{aligned}$$

Let us consider for example the first term,

$$\sum_{l=0}^{mt} r_t(l) f(k-l, s-1, n-s-t+1+q)$$

Since $r_t(l) = 0$ if $l \geq mt$, we observe that

$$\sum_{l=0}^{mt} r_t(l) f(\dots) = \sum_{l=0}^{\infty} r_t(l) f(\dots)$$

By the substitutions $\tau := t-2$, $\sigma := s-2$ and $\rho := n-3-\tau-\sigma$

$$\sum_{t=3}^{n-1} \sum_{s=2}^{n-t+1} (s-1) \binom{n-1-s}{t-2} \sum_{l=0}^{mt} r_t(l) f(k-l, s-1, n-s-t+1+q) =$$

$$\sum_{\substack{\tau \geq 1 \\ \sigma, \rho \geq 0 \\ \tau+\sigma+\rho=n-3}} (\sigma+1) \binom{\rho+\tau}{\tau} \sum_{l=0}^{\infty} r_{\tau+2}(l) f(k-l, \sigma+1, \rho+q)$$

Because

$$\frac{d}{dz} F(k,q) = \sum_{n \geq 0} (n+1) f(k, n+1, q) z^n$$

it follows that

$$\sum_{n \geq 0} \Big( \sum_{\substack{\tau \geq 1 \\ \sigma, \rho \geq 0 \\ \tau+\sigma+\rho=n-3}} (\sigma+1) \binom{\rho+\tau}{\tau} \sum_{l=0}^{\infty} r_{\tau+2}(l) f(k-l, \sigma+1, \rho+q) \Big) z^n$$

$$= \sum_{\substack{\tau \geq 1 \\ \sigma, \rho \geq 0 \\ \tau+\sigma+\rho \geq 3}} z^{\rho+\tau+3} \frac{d}{dz} \Big( \binom{\rho+\tau}{\tau} \sum_{l=0}^{\infty} r_{\tau+2}(l) F(k-l, \rho+q) \Big)$$

We make the same argument for the other terms and obtain

$$F(k,q) = z^2 \frac{d}{dz} \sum_{i=1}^{m-1} F(k-i,q) + zF(k,q+1) + \sum_{\substack{\sigma, \rho \geq 0 \\ \sigma+\rho \geq 3}} z^{\sigma+3} \frac{d}{dz} \Big( \sum_{i=0}^{m-1} F(k-i, \rho+q) \Big) +$$

$$\sum_{\substack{\tau \geq 1 \\ \sigma, \rho \geq 0 \\ \tau+\sigma+\rho \geq 3}} z^{\rho+\tau+3} \frac{d}{dz} \Big( \binom{\rho+\tau}{\tau} \sum_{l=0}^{\infty} \big( r_{\tau+2}(l) F(k-l, \rho+q) - r_{\tau+1}(l) F(k-l-m, \rho+q) +$$

$$r_\tau(l) F(k-l-m, \rho+q) - r_{\tau+1}(l) F(k-l, \rho+q) \big) \Big)$$

$\square$

## 3.3 Geometry of the exceptional divisors

If $C$ is a nonsingular algebraic curve, we obtain a modular description of the exceptional divisors in $C[n]$:

**Proposition 3.3.1.** *For each subset $S \subseteq \{1, \dots, n\}$ with at least two elements, the exceptional divisor $D(S)$ of $C[n]$ is isomorphic to the product*

$$C[n+1-s] \times \overline{\mathcal{M}}_{0,s+1}$$

*where $s = \#S$.*

*Proof.* The statement follows from the description of $D(S)$ by means of the language of screens (see §2 of [13]). Indeed, a point in $D(S)$ corresponds to a point in $C[n+1-s]$ and a sequence of screens for $S$ and its subsets $T \subset S$, successively separating the points labeled by $S$. Recall that in the general case of a non-singular algebraic variety $X$ of arbitrary dimension, a screen for $S$ at $x$ is defined to be the class of a labeled collection of points $x_a$ in $T_xX$, $a \in S$, not all equal. Two such families $\{x_a\}$ and $\{x'_a\}$ are equivalent if there exist $\lambda \in \mathbb{C}^*$ and $v \in T_xX$ such that $x_a = \lambda x'_a + v$. This equivalence relation means that collections of $S$ labeled points in $T_xX$ which differ by a translation or a homothety define the same screen. In other words, a screen for $S$ at $x$ is a point of $\mathbb{P}(T_x^S X / T_x X)$.

In the case $X = C$ is a curve, a screen $\{x_a\}$ determines $\#\{x_a\}$ distinct ordered points of $T_xC = \mathbb{C}$ up to an affinity of $\mathbb{C}$, hence we may associate to the screen $\{x_a\}$ a copy of $\mathbb{P}^1$ with three distinguished points $\{0, 1, \infty\}$ and $\#\{x_a\}$ marked points. We call special a marked point which appears more than once in $\{x_a\}$. All this said, we identify each point $p \in D(S)$ with the corresponding point in $C[n+1-s]$ and the stable curve in $\overline{\mathcal{M}}_{0,s+1}$ obtained from the sequence of screens for $p$ by taking all associated $\mathbb{P}^1$'s and joining any special point $x$ with the point $\infty$ of the $\mathbb{P}^1$ associated to the screen separating $x$. □

As a consequence, we determine the Chow ring of $D(S)$:

**Corollary 3.3.1.** *We have*

$$A^*(D(S)) = A^*(C[n+1-s]) \otimes A^*(\overline{\mathcal{M}}_{0,s+1}).$$

*More explicitely,*

$$A^*(D(S)) = A^*(C^{n+1-s})[D(T)]_{\substack{T \subset \{1,\dots,n+1-s\} \\ |T| \geq 2}} / I \otimes \mathbb{Z}[\delta_{0,A}]_{\substack{A \subset \{1,\dots,s+1\} \\ |A| \geq 3}} / J$$

*where the ideals of relations $I$ and $J$ are described in [13], Theorem 6, and [21], § 4, Theorem 1, respectively.*

*Proof.* Just apply Proposition 3.3.1 and [21], § 2, Theorem 2. □

## 3.4 The invariant Neron-Severi group

Let $S_n$ denote the symmetric group acting on $\{1, \ldots, n\}$ and let $C$ be a projective curve of genus $g \geq 1$ with general moduli. We are looking for an explicit description of $NS^{S_n}(C[n])$, the $S_n$-invariant part of the Neron-Severi group of $C[n]$. The first step is the following:

**Proposition 3.4.1.** *We have*

$$NS(C[n]) = (NS(C^n) \oplus \bigoplus_S \mathbb{Z}[D_S])/I,$$

*where $S \subseteq \{1, \ldots, n\}$, $|S| \geq 2$, and $I$ is the ideal generated by*

$$\sum_{S \supset \{a,b\}} D_S - \Delta_{\{a,b\}} \tag{3.18}$$

*for all $a, b$ such that $1 \leq a < b \leq n$.*

*Proof.* By [13], Theorem 6, we know that the Chow ring $A^*(C[n])$ is the polynomial ring $A^*(C^n)[D_S]/I$, where $\{D_S\}_S$ are $2^n - n - 1$ variables, each one corresponding to the class of the exceptional divisor $D(S)$, one for each subset $S$ of $\{1, \ldots, n\}$ with at least two elements, and $I$ is the ideal given by the following relations:

1. $D_S \cdot D_T$ if $S \cap T$ is a proper nonempty subset of $S$ and $T$
2. $J_S \cdot D_S$
3. $c_{a,b}(\sum_{S \supset \{a,b\}} D_S)$.

Here $J_S$ is the kernel of the restriction map from $A^*(C^n)$ to $A^*(\Delta_S)$, $c_{a,b}(t) \in A^*(C^n)[t]$ is the polynomial defined by

$$c_{a,b}(t) = -t + p^*_{a,b}([\Delta])$$

where $p_{a,b} : C^n \to C \times C$ is the $\{a, b\}$-projection corresponding to the $a$-th and the $b$-th factors, $[\Delta]$ is the class of the diagonal of $C \times C$ and $\Delta_S \subset C^n$ corresponds to the points with labels in $S$ equal. In particular, $A^1(C[n]) = (A^1(C^n) \oplus \bigoplus_S \mathbb{Z}[D_S])/I$ and in codimension one all relations of the ideal $I$ reduce to (3.18). □

We notice that the ideal $I$ generated by (3.18) is $S_n$-invariant. We also recall a couple of elementary facts:

**Lemma 3.4.1.** *If $B$ and $C$ are groups and $S_n$ acts on them and $A = B \oplus C$, then $A^{S_n} = B^{S_n} \oplus C^{S_n}$.*

**Lemma 3.4.2.** *If $A \xrightarrow{f} B$ is surjective and $f$ is $S_n$-invariant then $A^{S_n} \xrightarrow{f} B^{S_n}$ is surjective.*

*Proof.* For each $b \in B^{S_n}$ there exists $a \in A$ such that $f(a) = b$. If we introduce the $S_n$-invariant element

$$\bar{a} = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma(a),$$

then we have

$$f(\bar{a}) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma(a)) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(f(a)) = b.$$

□

As a consequence, we obtain the following:

**Proposition 3.4.2.** *The map*

$$NS(C^n)^{S_n} \oplus (\bigoplus_S \mathbb{Z}[D_S])^{S_n} \longrightarrow NS(C[n])^{S_n}$$

*is surjective, hence $NS(C[n])^{S_n}$ is generated by any set of generators of $NS(C^{(n)}) \oplus (\bigoplus_S \mathbb{Z}[D_S])^{S_n}$, where $C^{(n)}$ is the $n-$th symmetric product of $C$.*

The Neron-Severi group of $C^{(n)}$ can be described as follows (see §2 in [32]). Fix a point $x$ on $C$ and let:

$$\begin{array}{rcl} u_n : C^{(n)} & \rightarrow & J(C) \\ D & \mapsto & \mathcal{O}(D - nx), \end{array}$$

where $J(C)$ is the Jacobian variety of $C$, and

$$\begin{array}{rcl} i_{n-1} : C^{(n-1)} & \rightarrow & C^{(n)} \\ D & \mapsto & D + x. \end{array}$$

Let $\Theta_n$ be the pull-back under $u_n$ of the class in $NS(J(C))$ that corresponds to the theta divisor. Denote by $x$ the class of $i_{n-1}(C^{(n-1)})$ in $C^{(n)}$. Then on $C^{(n)}$ there are three natural divisors: $\Theta_n$, $x$ and $\delta$ corresponding to the big diagonal $\bigcup_{a,b} \Delta_{\{a,b\}}$. Since $C$ has general moduli, $NS(C^{(n)})$ is a 2-dimensional group and any two of the above three divisors provide a basis of it; a relation among them is given by

$$\frac{\delta}{2} = (n + g - 1)x - \Theta_n.$$

On the other hand, the variables $D_S$ relation (3.18) in $A^*(C[n])$. Summation over all possible pairs of distinct indices yields:

$$\delta = \sum_{a,b} p^*_{a,b}([\Delta]) = \sum_{s=2}^{n} \binom{n}{2} D[s],$$

where $D[s] = \sum_{|S|=s} D_S$ for $2 \le s \le n$. Hence we obtain the following:

**Corollary 3.4.1.** *The group $NS^{S_n}(C[n])$ is generated by the divisors $D[s]$ for $2 \le s \le n$ together with one more divisor chosen between $\Theta_n$ and $x$.*

## 3.5 Functorial description of the tangent space

In this section we give a functorial description of the points of the tangent space $TX[n]$ by means of the analogous description that Fulton and MacPherson made for $X[n]$ in Theorem 4 of [13]. In this case $X$ is defined over $\mathbb{C}$.

First we outline a description of the points of a blow-up and of the tangent space to that blow-up. The idea is that a tangent vector to a blow-up is the same as a tangent vector to the base together with a tangent vector in the direction of the exceptional divisor.

Let $M$ be a scheme. According to Grothendieck's definition, a $\mathbb{C}$-point of $M$, $x \in M(\mathbb{C})$, is the same as a morphism

$$Spec\mathbb{C} \xrightarrow{f_x} M$$

and a tangent vector, $v \in T_xM \subset M(\mathbb{C}[\epsilon])$, is a morphism $Spec\mathbb{C}[\epsilon] \xrightarrow{v} M$ which makes the diagram

$$\begin{tikzcd} Spec\mathbb{C} \arrow[r, "f_x"] \arrow[d] & M \arrow[d, equal] \\ Spec\mathbb{C}[\epsilon] \arrow[r, "v"] & M \end{tikzcd}$$

commutative.

Let now $Z \subset X$ be a closed regular embedding of varieties and consider $\pi : Bl_ZX \to X$. For each variety $Y$, a $Y$-point of $Bl_ZX$, $h : Y \to Bl_ZX$, is a $Y$-point of $X$, $f : Y \to X$, and a $Y$-point in the exceptional divisor $\mathbb{P}(N_ZX)$, $g : Y \to \mathbb{P}(N_ZX)$. By Proposition 7.12 pag.162 of [20], the last point is equivalent to a surjective morphism

$$g^*\pi^*I_Z = f^*I_Z \to \mathcal{L} \to 0$$

where $\mathcal{L}$ is an invertible sheaf on $Y$ defined up to isomorphisms. A closed point $\tilde{x}$ in $Bl_ZX$, that is a point in $Bl_ZX(\mathbb{C})$, is given by

$$\tilde{x} = (Spec\mathbb{C} \xrightarrow{f_x} X, I_Z(x) \overset{\varphi}{\twoheadrightarrow} \mathbb{C})$$

where $I_Z(x)$ is $f_x^*I_Z = I_{Z,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ and $\varphi$ is defined up to an isomorphism of $\mathbb{C}$.

Analogously, a tangent vector $\tilde{\tau} \in Bl_ZX(\mathbb{C}[\epsilon])$ is given by

$$\tilde{\tau} = (Spec\mathbb{C}[\epsilon] \xrightarrow{\tau} X, \tau^*I_Z \overset{\tilde{\varphi}}{\twoheadrightarrow} \mathbb{C}[\epsilon])$$

If $\tilde{\tau} \in Bl_ZX(\mathbb{C}[\epsilon])$ is a tangent vector at the point $\tilde{x}$, then we have compatibility between $\tau$ and $f_x$ and also between $\varphi$ and $\tilde{\varphi}$, that is the following diagram

$$\begin{tikzcd} \tau^*I_Z = I_{Z,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}[\epsilon] \arrow[r, "\tilde{\varphi}"] \arrow[d, "/\epsilon"'] & \mathbb{C}[\epsilon] \arrow[d, "/\epsilon"] \\ f_x^*I_Z = I_{Z,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \arrow[r, "\varphi"] & \mathbb{C} \end{tikzcd}$$

commutes.

We recall the first part of Theorem 4 of [13] and we write it in a suitable form to our aim. It provides a functorial description of the points of $X[n]$:

**Theorem 3.5.1.** *Each map $Spec\mathbb{C} \to X[n]$ is the datum of:*

1. *$x \in X^n$, that is $n$ morphisms $f_a : Spec\mathbb{C} \to X$, $a \in N$;*
2. *for each $S \subseteq N$, $|S| \geq 2$, $\varphi_S : I_{\Delta_S,x}/m_x I_{\Delta_S,x} \twoheadrightarrow \mathbb{C}$, up to non zero scalars;*
3. *the compatibility condition: $\forall T \subset S$, $\Delta_S \overset{i}{\hookrightarrow} \Delta_T$ induces $i_{TS} : I_{\Delta_T} \to I_{\Delta_S}$ and we require that there are (unique) $\gamma_{TS} \in \mathbb{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} I_{\Delta_T,x} & \xrightarrow{i_{TS}} & I_{\Delta_S,x} \\ \downarrow & & \downarrow \\ I_{\Delta_T,x}/m_x I_{\Delta_T,x} & \longrightarrow & I_{\Delta_S,x}/m_x I_{\Delta_S,x} \\ {\scriptstyle\varphi_T}\downarrow & & \downarrow{\scriptstyle\varphi_S} \\ \mathbb{C} & \xrightarrow{\gamma_{TS}} & \mathbb{C} \end{array}$$

Now we establish a functorial description of the points of the tangent space of a blow-up having a description of the points of the blow-up. Finally, with the functorial description of the points of $X[n]$, we find a description of $TX[n]$.

**Proposition 3.5.1.** *The tangent space of $Bl_Z X$ at the point $\tilde{x} = (f_x, \varphi)$ is the kernel of the following morphism*

$$\begin{aligned} Hom(m_x/m_x^2, \mathbb{C}) \oplus Hom(I_{Z,x}/m_x^2 I_{Z,x}, \mathbb{C})/\mathbb{C}\varphi_| &\longrightarrow Hom(m_x/m_x^2 \otimes I_{Z,x}/m_x I_{Z,x}, \mathbb{C}) \\ (\lambda, \phi) &\longmapsto \phi_| - \lambda\varphi \end{aligned}$$

*where $\phi_|$ is defined to be*

$$\phi_| : m_x/m_x^2 \otimes I_{Z,x}/m_x I_{Z,x} \overset{\cdot}{\longrightarrow} I_{Z,x}/m_x^2 I_{Z,x} \xrightarrow{\phi} \mathbb{C}$$

*and $\varphi_| : I_{Z,x}/m_x^2 I_{Z,x} \to I_{Z,x}/m_x I_{Z,x} \xrightarrow{\varphi} \mathbb{C}$.*

*Proof.* Observe that a tangent vector $v \in T_x X$, $v : Spec\mathbb{C}[\epsilon] \to X$, is the same as

$$\lambda : m_x/m_x^2 \to \mathbb{C}$$

(see for istance ex.II 2.8 of [20]). A tangent vector in $Bl_Z X$ at a point $\tilde{x} = (f_x, \varphi)$ is given by the pair $(\lambda, \tilde{\varphi})$ defined up to an isomorphism of $\mathbb{C}[\varepsilon]$ and such that $\tilde{\varphi}$ restricts to $\varphi$. The datum $\tilde{\varphi}$ is equivalent to have $\hat{\tilde{\varphi}}$

$$\hat{\tilde{\varphi}} : I_{Z,x} \to I_{Z,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}[\epsilon] \overset{\tilde{\varphi}}{\twoheadrightarrow} \mathbb{C}[\epsilon]$$

Because of $\tilde{\varphi}$ restricts to $\varphi$

$$\hat{\tilde{\varphi}} = \varphi + \epsilon\psi$$

where

$$\psi : I_{Z,x} \to \mathbb{C}$$

is a $\mathbb{C}$-linear map such that $\varphi + \epsilon\psi$ is $\mathcal{O}_{X,x}$-linear. This last condition ca be reformulated in the following way. For each $g \in I_{Z,x}$ and $h \in \mathcal{O}_{X,x}$

$$(\varphi + \epsilon\psi)(h \cdot g) = c \cdot \varphi(g) + \epsilon(c \cdot \psi(g) + \psi(f \cdot g))$$

equals

$$v^*(h) \cdot (\varphi + \epsilon\psi)(g) = (c + \epsilon\lambda(f))(\varphi(g) + \epsilon\psi(g))$$

where $h = c + f$, with $c \in \mathbb{C}$ and $f \in m_x$. The equality is equivalent to

$$\psi(f \cdot g) = \lambda(f)\varphi(g)$$

In particular $\psi$ factors because $\lambda(f^2) = 0$

$$\psi : I_{Z,x} \to I_{Z,x}/m_x^2 I_{Z,x} \xrightarrow{\phi} \mathbb{C}$$

Now it remains to express the isomorphism of $\mathbb{C}[\epsilon]$. If we consider $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ (resp. $\hat{\tilde{\varphi}}_1$ and $\hat{\tilde{\varphi}}_2$) then there exists an isomorphism of $\mathbb{C}[\epsilon]$ such that

$$\begin{array}{ccc} I_{Z,x} & \xrightarrow{\hat{\tilde{\varphi}}_1} & \mathbb{C}[\epsilon] \\ \| & & \downarrow \cong \\ I_{Z,x} & \xrightarrow{\hat{\tilde{\varphi}}_2} & \mathbb{C}[\epsilon] \end{array}$$

commutes. Since $\hat{\tilde{\varphi}}_1 = (\varphi_1, \epsilon\psi_1)$ and the isomorphism of $\mathbb{C}[\epsilon]$ is given by a matrix

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$$

with $a \in \mathbb{C}^*$. Fixing $a$ is the same as fixing a representative of $\varphi_1 : I_Z(x) \to \mathbb{C}$, so we choose $a = 1$. It follows then that $\psi_1$ can be modified by $b\varphi_1 + \psi_1$, with $b \in \mathbb{C}$ and we are done. $\square$

**Theorem 3.5.2.** *In the notation of Proposition 3.5.1, the tangent space of $X[n]$ at the point $\tilde{x} = (f_x, \varphi)$ is the kernel of the morphism from*

$$Hom(m_x/m_x^2, \mathbb{C}) \oplus \bigoplus_S \left(Hom(I_{\Delta_S,x}/m_x^2 I_{\Delta_S,x}, \mathbb{C})/\mathbb{C}\varphi_{S|} \oplus \bigoplus_{T\subset S} \mathbb{C}\right)$$

*to*

$$\bigoplus_S \left(Hom(m_x/m_x^2 \otimes I_{\Delta_S,x}/m_x I_{\Delta_S,x}, \mathbb{C}) \oplus \bigoplus_{T\subset S} Hom(I_{\Delta_T,x}, \mathbb{C})\right)$$

*defined by*

$$(\lambda; \ldots, \phi_S, \ldots; b_{TS}) \quad \longmapsto (\ldots, \phi_{S|} - \lambda\varphi_S, \ldots; \ldots, b_{TS}\hat{\varphi}_T + \gamma_{TS}\hat{\psi}_T - \hat{\psi}_S \circ i_{TS}, \ldots)$$

*Proof.* First apply Proposition 3.5.1 at each $\varphi_S : I_{\Delta_S,x}/m_x I_{\Delta_S,x} \twoheadrightarrow \mathbb{C}$, for all $S \subseteq N$, $|S| \geq 2$. It remains then to express the compatibility condition for each $T \subset S$. This condition is the commutativity of the following diagram

$$\begin{tikzcd}
I_{\Delta_T,x} \arrow[r, "i_{TS}"] \arrow[d] & I_{\Delta_S,x} \arrow[d] \\
(I_{\Delta_T,x} \otimes \mathbb{C}) \oplus (I_{\Delta_T,x} \otimes \epsilon\mathbb{C}) \arrow[r, "i_{TS}\otimes id_{\mathbb{C}[\epsilon]}"] \arrow[d, "\varphi_T+\epsilon\psi_T"'] & (I_{\Delta_S,x} \otimes \mathbb{C}) \oplus (I_{\Delta_S,x} \otimes \epsilon\mathbb{C}) \arrow[d, "\varphi_S+\epsilon\psi_S"] \\
\mathbb{C} \oplus \epsilon\mathbb{C} \arrow[r, "\begin{pmatrix} a_{TS} & 0 \\ b_{TS} & a_{TS} \end{pmatrix}"] & \mathbb{C} \oplus \epsilon\mathbb{C}
\end{tikzcd}$$

For each $f \in I_{\Delta_T,x}$ we obtain the conditions

$$\begin{aligned}
&a_{TS}\varphi_T(f \otimes 1) = \varphi_S(i_{TS}(f) \otimes 1) \\
&b_{TS}\varphi_T(f \otimes 1) + a_{TS}\psi_T(f \otimes 1) = \psi_S(i_{TS}(f) \otimes 1)
\end{aligned}$$

By uniqueness of $\gamma_{TS}$ it follows $a_{TS} = \gamma_{TS}$. Therefore the condition reads

$$b_{TS}\hat{\varphi}_T + \gamma_{TS}\hat{\psi}_T - \hat{\psi}_S \circ i_{TS} = 0$$

If $i_{TS}$ is an inclusion, this is a linear condition in $\varphi, \psi$'s. □

An easy calculation shows that for each triple $T \subset S \subset F$

$$\gamma_{SF}\gamma_{TS} = \gamma_{TF}$$

and we deduce the following relation

$$b_{TF} = b_{SF}\gamma_{TS} + \gamma_{SF}b_{TS}$$

# Chapter 4

# The Chow groups of $\mathbb{C}^m[n]$

In Theorem 6 of [13] Fulton and MacPherson express the intersection ring of $X[n]$ as a polynomial ring with coefficients in the intersection ring of $X^n$. However, in such a presentation generators and relations of the single Chow groups of $X[n]$ are only implicit and difficult to read out.

In this chapter we focus on the case $X = \mathbb{C}^m$ and try to find the Chow groups of $\mathbb{C}^m[n]$.

For $n = 1, 2, 3, 4$ we are able to determine bases of these groups and in particular their dimension (see Propositions (4.1.1), (4.2.1), (4.3.1), (4.3.2) and (4.3.3)).

From these results we deduce a symmetry of the groups with respect to the codimension $[\frac{(n-1)m-1}{2}]$ (see Theorems (4.1.1), (4.2.1) and (4.3.1)). If $n = 1, 2, 3$ we are able to realize such symmetry through explicit isomorphisms which are obtained by intersecting with cycles expressed by the exceptional divisors (see Theorems (4.1.1), (4.2.1)). Unfortunately, we were not able to generalize such isomorphisms even to the case $n = 4$.

From our results in the cases $n = 1, 2, 3, 4$ we are led to formulate the following conjecture:

Conjecture *1.* There exists a symmetry between the Chow groups of $\mathbb{C}^m[n]$ with respect to the codimension $[\frac{(n-1)m-1}{2}]$, that is

$$\dim A^p(\mathbb{C}^m[n]) = \dim A^{(n-1)m-1-p}(\mathbb{C}^m[n])$$

for all $p \leq [\frac{(n-1)m-1}{2}]$. In particular $A^p(\mathbb{C}^m[n]) = 0$ if $p \geq (n-1)m$.

A strong motivation for this conjecture is that it holds for $n = 4$. In fact the configuration space $X^n \setminus \bigcup \Delta_{a,b}$ can be compactified in different ways in the sense of De Concini and Procesi who developed a general approach to compactify complements of linear subspace arrangements by iterated blow-ups (see [9]). From this point of view, the Fulton-MacPherson is the minimal compactification whereas the polydiagonal one, studied by Ulyanov in [35], is the maximal. These two are isomorphic for $n = 2, 3$ whereas they are different for $n \geq 4$, thus the case $n = 4$ is the critical one.

To be bolder, we also propose the following

Conjecture *2.* There exist isomorphisms which realize the symmetry mentioned above and they can be expressed as intersection products with cycles generated by intersections of the exceptional divisors.

- **The general setting**

We recall Theorem 6 of [13] for the formula for the Chow ring of $X[n]$ when $X$ is a non-singular algebraic variety of dimension $m$

$$A^*(X[n]) = A^*(X^n)[D_S]/I$$

where $I$ is the ideal given by the elements

1. $D_S \cdot D_T$, if $S$ and $T$ "overlap" (i.e. $\emptyset \neq S \cap T \subsetneq S, T$)
2. $J_S \cdot D_S$
3. $c_{a,b}(\sum_{S \ni a,b} D_S)$, $1 \leq a < b \leq n$

where $J_S$ is the kernel of the restriction map $A^*(X^n) \to A^*(\Delta_S)$ and $c_{a,b}(t)$ is the polynomial in $A^*(X^n)[t]$ defined by

$$c_{a,b}(t) = \sum_{i=1}^{m} (-1)^i p_a^*(c_{m-i}) t^i + p_{a,b}^*([\Delta])$$

where $[\Delta] \in A^m(X \times X)$ and $p_a, p_{a,b}$ are the two projections $p_a : X^n \to X_{(a)}$, $p_{a,b} : X^n \to X_{(a)} \times X_{(b)}$.

- If $X = \mathbb{C}^m$ then the Chow ring of $\mathbb{C}^m[n]$ is

$$A^*(\mathbb{C}^m[n]) = \bigoplus_S \mathbb{Z}[D_S]/I$$

and

$$\begin{aligned} &0 = J_S = ker(A^*(X^n) = \mathbb{Z} \to A^*(\Delta_S) = \mathbb{Z}) \\ &A^m(X \times X) = 0 \\ &c_{m-i} = c_{m-i}(TX) \cap X = \delta_{i,m}[X] \end{aligned}$$

so the elements of the second type of the ideal $I$ are zero in $A^*(X^n)[D_S]$ and the Chow polynomials reduce to

$$c_{a,b}(\sum_{S \ni a,b} D_S) = (-1)^m [\mathbb{C}^{nm}] (\sum_{S \ni a,b} D_S)^m$$

$\forall\ 1 \leq a < b \leq n$.

Since $\mathbb{C}^m[n]$ is irreducible

$$A_{nm}(\mathbb{C}^m[n]) = A^0(\mathbb{C}^m[n]) = \mathbb{Z}[\mathbb{C}^m[n]]$$

for all $n, m$.

If $n = 1$ then $\mathbb{C}^m[1] = \mathbb{C}^m$ and we already know that $A^0(\mathbb{C}^m) = \mathbb{Z}[\mathbb{C}^m]$ and the other groups are zero.

For the moment we consider $m \geq 2$ and because there are no relations of degree one in the divisors for $m > 1$ we have

$$A_{nm-1}(\mathbb{C}^m[n]) = \bigoplus_S \mathbb{Z}[D_S]$$

We therefore study only the cycles of codimension $\geq 1$, even if the propositions and the theorems continue to be valid in the case $m = 1$, indeed terms depending on a non-defined index must be considered as zero.

**Notation 4.0.1.** In the sequel we shall indicate with $\mathscr{B}_{p,m,n}$ a basis of $A^p(\mathbb{C}^m[n])$. In addition, we will make an abuse of notation by writing $D_S$ instead of its class $[D_S]$ modulo rational equivalence.

## 4.1 The case $\mathbb{C}^m[2]$

Tha study of this situation is rather easy because there is only one exceptional divisor, $D_{12}$, satisfying the unique condition $D_{12}^m = 0$ and we are able to give a basis of the Chow groups.

For later applications we point out two facts which in this case are obvious.

**Proposition 4.1.1.** *The Chow groups of $\mathbb{C}^m[2]$ can be realized as*

$$A^p(\mathbb{C}^m[2]) = \mathbb{Z}[D_{12}^p]$$

*for all $1 \leq p \leq m - 1$.*

Moreover there are isomorphisms between the Chow groups:

**Theorem 4.1.1.** *The non-zero map*

$$\cdot[D_{12}^{m-1-2p}] : A^p(\mathbb{C}^m[2]) \to A^{m-1-p}(\mathbb{C}^m[2])$$

*is an isomorphism for $0 \leq p \leq [\frac{m-1}{2}]$.*

## 4.2 The case $\mathbb{C}^m[3]$

In oredr to understand the case $n = 3$ we first analyse in detail the situation for $m = 2$ in (a) and then in (b) we will treat $m \geq 2$.

(a) $\mathbb{C}^2[3]$

When $m = 2, n = 3$ the relations of $I$ are

$$D_{12} \cdot D_{13} = D_{12} \cdot D_{23} = D_{13} \cdot D_{23} = 0 \tag{4.1}$$

and

$$\begin{aligned} D_{12}^2 + D_{123}^2 + 2D_{12} \cdot D_{123} = 0 \\ D_{13}^2 + D_{123}^2 + 2D_{13} \cdot D_{123} = 0 \\ D_{23}^2 + D_{123}^2 + 2D_{23} \cdot D_{123} = 0 \end{aligned} \tag{4.2}$$

We have

$$A_5(\mathbb{C}^2[3]) = \mathbb{Z}[D_{12}] \oplus \mathbb{Z}[D_{13}] \oplus \mathbb{Z}[D_{23}] \oplus \mathbb{Z}[D_{123}]$$

The 4-cycles are generated by the quadratic forms $D_{123}^2, D_{12}^2, D_{13}^2, D_{23}^2, D_{12} \cdot D_{123}, D_{13} \cdot D_{123}, D_{23} \cdot D_{123}$ (the other products are zero). Because of the three relations in (4.2), each $D_{ij}^2$ can be expressed as a sum of the $D_S \cdot D_{123}$'s which are independent and therefore

$$A_4(\mathbb{C}^2[3]) = \mathbb{Z}[D_{12} \cdot D_{123}] \oplus \mathbb{Z}[D_{13} \cdot D_{123}] \oplus \mathbb{Z}[D_{23} \cdot D_{123}] \oplus \mathbb{Z}[D_{123}^2]$$

The 3-cycles can be obtained intersecting the elements of the basis of the 4-cycles with the exceptional divisors $D_S$. Bearing the relations in (4.1) in mind, we choose the elements of the basis among the following products

$$\begin{aligned} &D_{ij} \cdot D_{123}^2 \\ &D_{ij}^2 \cdot D_{123} \\ &D_{123}^3 \end{aligned}$$

for $1 \leq i, j \leq 3$. By using both the relations (4.1) and (4.2) we see that the 3-cycles of the type $D_{ij} \cdot D_{123}^2$ are zero in the quotient indeed $D_{123}^2$ can be expressed as a sum of $D_{hk}^2, D_{hk} \cdot D_{123}$ and for example we have

$$-D_{12} \cdot D_{123}^2 = D_{12} \cdot (D_{13}^2 + 2D_{13} \cdot D_{123}) = D_{12} \cdot D_{13}^2 + 2D_{12} \cdot D_{13} \cdot D_{123} = 0 \tag{4.3}$$

the same holds for the other two couples of indexes.
We now look at the elements of the type $D_{ij}^2 \cdot D_{123}$ and take for example $D_{12}^2 \cdot D_{123}$. In this case by (4.2) we can substitute $D_{12}^2 \cdot D_{123}$ by a sum in $D_{123}^3$ and $D_{12} \cdot D_{123}^2$, the latter vanishes from the previous argument and we have no other condition. Therefore

$$A_3(\mathbb{C}^2[3]) = \mathbb{Z}[D_{123}^3]$$

In order to find the 2-cycles, we intersect the generator $[D_{123}^3]$ of the 3-cycles with all the $D_S$'s.
From (4.3) $D_{ij} \cdot D_{123}^3 = 0$. Now let observe that

$$D_{123}^4 = D_{123}^2 \cdot D_{123}^2 = (D_{12}^2 + 2D_{12} \cdot D_{123}) \cdot (D_{13}^2 + 2D_{13} \cdot D_{123}) = 0$$

so the first Chow group which vanishes is

$$A_2(\mathbb{C}^2[3]) = 0$$

We can summarize the preceding calculation in the following tabular:

| $k$-cycles | basis of $A_k(\mathbb{C}^2[3])$ | dim of $A_k(\mathbb{C}^2[3])$ |
|---|---|---|
| 6 | $\mathbb{C}^2[3]$ | 1 |
| 5 | $D_{12}, D_{13}, D_{23}, D_{123}$ | 4 |
| 4 | $D_{12} \cdot D_{123}, D_{13} \cdot D_{123}, D_{23} \cdot D_{123}, D_{123}^2$ | 4 |
| 3 | $D_{123}^3$ | 1 |
| 2 | 0 | 0 |

(b) $\mathbb{C}^m[3]$

In this case the ideal $I$ is generated by the relations in (4.1) together with the following three others

$$\begin{aligned}
(D_{12} + D_{123})^m &= \sum_{h=0}^{m} \binom{m}{h} D_{12}^h \cdot D_{123}^{m-h} = 0 \\
(D_{13} + D_{123})^m &= \sum_{h=0}^{m} \binom{m}{h} D_{13}^h \cdot D_{123}^{m-h} = 0 \\
(D_{23} + D_{123})^m &= \sum_{h=0}^{m} \binom{m}{h} D_{23}^h \cdot D_{123}^{m-h} = 0
\end{aligned} \tag{4.2$'$}$$

Since the relations in (4.2′) are in codim $= m$, the Chow groups $A_{2m+q}(\mathbb{C}^m[3]) = A^{m-q}(\mathbb{C}^m[3])$ are

$$\begin{aligned}
&A^{m-q}(\mathbb{C}^m[3]) \\
&= \bigoplus_{a=1}^{m-q} \mathbb{Z}[D_{12}^a \cdot D_{123}^{m-q-a}] \oplus \bigoplus_{b=1}^{m-q} \mathbb{Z}[D_{13}^b \cdot D_{123}^{m-q-b}] \oplus \bigoplus_{c=1}^{m-q} \mathbb{Z}[D_{23}^c \cdot D_{123}^{m-q-c}] \oplus \mathbb{Z}[D_{123}^{m-q}]
\end{aligned} \tag{4.4}$$

for all $1 \leq q \leq m-1$.

The dimension of such groups is therefore

$$\dim A^{m-q}(\mathbb{C}^m[3]) = 3(m-q) + 1 \tag{4.5}$$

for all $1 \leq q \leq m$. The elements of the basis of $A^{m+q}$, with $0 \leq q$, are products of the following type:

$$D_{ij}^x \cdot D_{123}^{m+q-x} \quad 0 \leq x \leq m+q, \quad 1 \leq i, j \leq 3$$

First we will study the case $q \leq m-1$.

Similarly to the case of $m = 2$, $D_{123}^m$ can be expressed as a sum of terms like $D_{ij}^h \cdot D_{123}^{m-h}$ with $h \geq 1$ and some fixed $i, j$. Therefore, by virtue of (4.1), we see that if $1 \leq a \leq q$

$$D_{12}^a \cdot D_{123}^{m+q-a} = -\sum_{h=1}^{m} \binom{m}{h} D_{13}^h \cdot D_{123}^{m-h} \cdot D_{12}^a \cdot D_{123}^{q-a} = 0$$

and we can eliminate all the following products for $1 \leq a \leq q$

$$\begin{aligned} D_{12}^a \cdot D_{123}^{m+q-a} &= 0 \\ D_{13}^a \cdot D_{123}^{m+q-a} &= 0 \\ D_{23}^a \cdot D_{123}^{m+q-a} &= 0 \end{aligned} \tag{4.3$'$}$$

In addition the terms of the form $D_{12}^a \cdot D_{123}^{m+q-a}$ with $m \leq a \leq m+q$, depend on some other products because by (4.2$'$) the factor $D_{12}^m$ can be expressed in function of the products $D_{12}^h \cdot D_{123}^{m-h}$, $h \geq 1$.

Now we are able to give a basis of the group $A^{m+q}(\mathbb{C}^m[3])$, $0 \leq q \leq m-1$:

$$\mathscr{B}_{m+q,m,3} = \left\{ \begin{array}{ll} D_{12}^a \cdot D_{123}^{m+q-a}, & q+1 \leq a \leq m-1; \\ D_{13}^b \cdot D_{123}^{m+q-b}, & q+1 \leq b \leq m-1; \\ D_{23}^c \cdot D_{123}^{m+q-c}, & q+1 \leq c \leq m-1; \\ D_{123}^{m+q} & \end{array} \right\} \tag{4.6}$$

and in these cases the dimension of the group is

$$\dim A^{m+q}(\mathbb{C}^m[3]) = 3(m-1-q)+1 = 3(m-q)-2 \qquad 0 \leq q \leq m-1 \tag{4.7}$$

If $q = m$, we get a basis of $A^{2m}(\mathbb{C}^m[3])$ intersecting the basis of $A^{2m-1}(\mathbb{C}^m[3])$ with the divisors $D_S$.

From above

$$A^{2m-1}(\mathbb{C}^m[3]) = \mathbb{Z}[D_{123}^{2m-1}] \tag{4.8}$$

The intersections $D_{ij} \cdot D_{123}^{2m-1}$ vanish because of (4.3$'$) and

$$D_{123}^{2m} = \sum_{h=1}^{m} \binom{m}{h} D_{12}^h \cdot D_{123}^{m-h} \cdot \sum_{k=1}^{m} \binom{m}{k} D_{13}^k \cdot D_{123}^{m-k} = 0$$

therefore

$$A^{m+q}(\mathbb{C}^m[3]) = 0 \qquad \forall q \geq m \tag{4.9}$$

and we can write the following tabular:

| $p$-cocycles | basis of $A^p(\mathbb{C}^m[3])$ | dim of $A^p(\mathbb{C}^m[3])$ |
|---|---|---|
| 0 | $\mathbb{C}^m[3]$ | 1 |
| 1 | $D_{12}, D_{13}, D_{23}, D_{123}$ | 4 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $m-q$ | $D_{123}^{m-q}$ <br> $D_{ij}^a \cdot D_{123}^{m-q-a}, 1 \leq i,j \leq 3, 1 \leq a \leq m-q$ | $3(m-q)+1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $m-1$ | $D_{123}^{m-1}$ <br> $D_{ij}^a \cdot D_{123}^{m-1-a}, 1 \leq i,j \leq 3, 1 \leq a \leq m-1$ | $3m-2$ |
| $m$ | $D_{123}^{m}$ <br> $D_{ij}^a \cdot D_{123}^{m-a}, 1 \leq i,j \leq 3, 1 \leq a \leq m-1$ | $3m-2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $m+q$ | $D_{123}^{m+q}$ <br> $D_{ij}^a \cdot D_{123}^{m+q-a}, 1 \leq i,j \leq 3, q+1 \leq a \leq m-1$ | $3(m-q)-2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $2m-2$ | $D_{123}^{2m-2}$ <br> $D_{12}^{m-1} \cdot D_{123}^{m-1}, D_{13}^{m-1} \cdot D_{123}^{m-1}, D_{23}^{m-1} \cdot D_{123}^{m-1}$ | 4 |
| $2m-1$ | $D_{123}^{2m-1}$ | 1 |
| $2m$ | 0 | 0 |

Now we are able to state the following:

**Proposition 4.2.1.** *The Chow groups of $X = \mathbb{C}^m[3]$ are given by the following expressions:*

$$A^p(X) = \bigoplus_{a=1}^{p} \mathbb{Z}[D_{12}^a \cdot D_{123}^{p-a}] \oplus \bigoplus_{b=1}^{p} \mathbb{Z}[D_{13}^b \cdot D_{123}^{p-b}] \oplus \bigoplus_{c=1}^{p} \mathbb{Z}[D_{23}^c \cdot D_{123}^{p-c}] \oplus \mathbb{Z}[D_{123}^p]$$

*for $1 \leq p \leq m-1$ and*

$$\begin{aligned} &A^p(X) \\ &= \bigoplus_{a=p-m+1}^{m-1} \mathbb{Z}[D_{12}^a \cdot D_{123}^{p-a}] \oplus \bigoplus_{b=p-m+1}^{m-1} \mathbb{Z}[D_{13}^b \cdot D_{123}^{p-b}] \oplus \bigoplus_{c=p-m+1}^{m-1} \mathbb{Z}[D_{23}^c \cdot D_{123}^{p-c}] \oplus \mathbb{Z}[D_{123}^p] \end{aligned}$$

*for $m \leq p \leq 2m-1$. When $p \geq 2m$ then $A^p(X) = 0$. In particular* $\dim A^p(X)$ *is $3p+1$, $2m-p-2$ and $0$ respectively.*

*Proof.* By formulae (4.4) and (4.5) we deduce the statement for codimension $p$ with $0 \leq p \leq m-1$, by (4.6), (4.7), and (4.8) we have it for $m \leq p \leq 2m-1$ and finally (4.9) gives the case $p \geq 2m$. $\square$

Moreover in this case we are able to determine an explicit isomorphism between the Chow groups which expresses the symmetry shown by the tabular above

**Theorem 4.2.1.** *The non-zero map defined by the following intersection product*

$$\cdot\Big(\sum_{ij}[D_{ij}^{m-p-1}\cdot D_{123}^{m-p}]+[D_{123}^{2m-1-2p}]\Big): A^p(\mathbb{C}^m[3])\to A^{2m-1-p}(\mathbb{C}^m[3])$$

*is an isomorphism for each* $0\le p\le m-1$.

*Proof.* In order to prove the isomorphism we will examine the action of such a product on the elements of the basis of $A^p(\mathbb{C}^m[3])$ for $0\le p\le m-1$

$$\mathscr{B}_{p,m,3}=\{D_{ij}^x\cdot D_{123}^{p-x}, 1\le x\le p; D_{123}^p\}_{ij}$$

with respect to the elements of the basis of the corresponding group $A^{2m-1-p}(\mathbb{C}^m[3])$

$$\mathscr{B}_{2m-1-p,m,3}=\{D_{ij}^y\cdot D_{123}^{2m-1-p-y}, m-p\le y\le m-1; D_{123}^{2m-1-p}\}_{ij}$$

The image of $D_{123}^p$ is

$$D_{123}^p\cdot\Big(\sum_{ij}D_{ij}^{m-p-1}\cdot D_{123}^{m-p}+D_{123}^{2m-1-2p}\Big)=\sum_{ij}D_{ij}^{m-p-1}\cdot D_{123}^{m}+D_{123}^{2m-1-p}=D_{123}^{2m-1-p}$$

The last equality is due to (4.3′) because, when there is the term $D_{ij}^{m-p-1}$, $1\le m-p-1=(2m-1-p)-m$.

Moreover, for all $\{i,j\}\in\{1,2,3\}$ and $1\le x\le p$

$$D_{ij}^x\cdot D_{123}^{p-x}\cdot\Big(\sum_{ij}D_{ij}^{m-p-1}\cdot D_{123}^{m-p}+D_{123}^{2m-1-2p}\Big)=D_{ij}^{x+m-p-1}\cdot D_{123}^{m-x}+D_{ij}^x\cdot D_{123}^{2m-1-p-x}$$

Since $1\le x\le p$ it follows that $m-p\le x+m-p-1\le m-1$ and therefore $D_{ij}^{x+m-p-1}\cdot D_{123}^{m-x}$ belongs to the basis $\mathscr{B}_{2m-1-p,m,3}$.

The dependent elements in $A^{2m-1-p}(\mathbb{C}^m[3])$ are

$$\begin{aligned} &D_{ij}^a\cdot D_{123}^{2m-1-p-a}=0, && 1\le a\le m-1-p\\ &D_{ij}^b\cdot D_{123}^{2m-1-p-b}, && m\le b\le 2m-1-p\end{aligned}$$

and as $p<m$ we will determine only when $1\le x\le p$ is in $\{1,\dots,m-1-p\}$.

If $p\le\frac{m-1}{2}$, that is if $p\le m-1-p$, then $D_{ij}^x\cdot D_{123}^{2m-1-p-x}$ vanishes for all $x$ and the matrix associated to the map $\cdot\big(\sum_{ij}[D_{ij}^{m-p-1}\cdot D_{123}^{m-p}]+[D_{123}^{2m-1-2p}]\big)$ is the unit matrix and therefore it is an isomorphism.

If $p>\frac{m-1}{2}$, that is if $p>m-1-p$, then the term $D_{ij}^x\cdot D_{123}^{2m-1-p-x}$ is zero when $1\le x\le m-p-1$, otherwise it is an element of the basis $\mathscr{B}_{2m-1-p,m,3}$. Therefore the matrix representing the map in the bases $\mathscr{B}_{p,m,3}$ and $\mathscr{B}_{2m-1-p,m,3}$ is

$$M=\left(\begin{array}{c|c} I_{3(m-1-p)} & A\\ \hline 0_{3(2p-m+1)+1,3(m-1-p)} & B\end{array}\right)$$

where $A$ is the following matrix of order $3(m-p-1) \times (3(2p-m+1)+1)$

$$A = \left(\begin{array}{ccccccc|c}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0_{3(m-p-1),3(3p-2m+2)+1} \\
0 & 0 & 0 & \ddots & 1 & 0 & 0 & \\
0 & 0 & 0 & \ddots & 0 & 1 & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 &
\end{array}\right)$$

and $B$ is the upper triangular square matrix of order $3(2p-m+1)+1$

$$B = \left(\begin{array}{cccccccc|c}
1 & 0 & \cdots & 0 & \overbrace{1 \quad\; 0}^{3(3p-2m+2)} & & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{array}\right)$$

Since $\det M = \det I_{3(m-1-p)} \det B = 1$, the map $\cdot\big(\sum_{ij}[D_{ij}^{m-p-1} \cdot D_{123}^{m-p}] + [D_{123}^{2m-1-2p}]\big)$ is invertible. □

## 4.3 The case $\mathbb{C}^m[4]$

In this section we first provide a basis and therefore the dimension of $A^p(\mathbb{C}^m[4])$ when there are no relations among the exceptional divisors, that is when $0 \leq p < m$. Afterwards we study in (a) the case $m = 2$ in detail and deduce a general method to be applied if $m \geq 2$ in (b).

Then we will see that we the cases $m \leq p \leq 2m-1$, $2m \leq p \leq 3m-1$ and $p \geq 3m-1$ have to be treated separately and in each case we will determine the number of generators of $A^p(\mathbb{C}^m[4])$ which are redundant in the virtual bases (4.10).

Now things start getting more complicated. First of all there are 11 exceptional divisors which constitute a basis of $A^1(\mathbb{C}^m[4])$

$$\begin{aligned}
&D_{12}, D_{13}, D_{14}, D_{23}, D_{24}, D_{34}\\
&D_{123}, D_{124}, D_{134}, D_{234}\\
&D_{1234}
\end{aligned}$$

and the products

$$D_{12} \cdot D_{34}, D_{13} \cdot D_{24}, D_{14} \cdot D_{23}$$

$$D_{ij}^2, D_{ijk}^2, D_{1234}^2$$

$$D_{ij} \cdot D_{ijk}, D_{ij} \cdot D_{1234}, D_{ijk} \cdot D_{1234}$$

do not vanish. Indeed

$$D_S \cdot D_T = 0 \qquad \forall\ S, T \subseteq \{1,2,3,4\}, \emptyset \neq S \cap T \subsetneq S, T \tag{4.1''}$$

The relations in codimension $m$ take the form

$$(D_{ij} + D_{ijh} + D_{ijk} + D_{1234})^m = 0 \qquad 1 \leq i, j \leq 4 \tag{4.2''}$$

where the four indexes are all distinct and $\{i, j, h, k\} = \{1, 2, 3, 4\}$.
Due to (4.1″) the generators of the groups $A^p(\mathbb{C}^m[4])$ are chosen among the elements of the following two types

$$\begin{aligned}
&D_{ij}^a \cdot D_{hk}^b \cdot D_{1234}^c\\
&D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^c \qquad a + b + c = p
\end{aligned}$$

Until $p < m$, these elements are all independent and so we can provide directly a basis of $A^p(\mathbb{C}^m[4])$

$$\mathscr{B}_{p,m,4} = \left\{ \begin{array}{lr}
D_{1234}^p; & \\
D_{ij}^a \cdot D_{1234}^{p-a}, & 1 \leq a \leq p;\\
D_{ijh}^a \cdot D_{1234}^{p-a}, & 1 \leq a \leq p;\\
D_{ij}^a \cdot D_{hk}^b \cdot D_{1234}^{p-a-b}, & 1 \leq a < b \leq p-1,\ 3 \leq a+b \leq p;\\
D_{ij}^a \cdot D_{hk}^a \cdot D_{1234}^{p-2a}, & 1 \leq a \leq [\frac{p}{2}];\\
D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^{p-a-b}, & 1 \leq a, b,\ 2 \leq a+b \leq p
\end{array} \right\}_{i,j,h,k} \tag{4.10}$$

**Proposition 4.3.1.** *When the codimension $p$ is $0 \leq p < m$, a basis of the Chow groups $A^p(\mathbb{C}^m[4])$ is (4.10) and their dimension is given by*

$$\dim A^p(\mathbb{C}^m[4]) = 1 + \frac{5}{2}\, p(1 + 3p) \tag{4.11}$$

*Proof.* The dimension of these groups is given by the following expression

$$\begin{aligned}\dim A^p(\mathbb{C}^m[4]) = 1 &+ \binom{4}{2}p + \binom{4}{3}p + \frac{1}{2}\binom{4}{2}\left[\frac{p}{2}\right] \\ &+ \binom{4}{2}\sum_{n=3}^{p} \#\Big\{(a,b) : a,b \in \{1,\dots,p-1\}, a<b, a+b=n\Big\} \\ &+ 2\binom{4}{2}\sum_{n=2}^{p} \#\Big\{(a,b) : a,b \in \{1,\dots,p-1\}, a+b=n\Big\}\end{aligned}$$

We make use of some notations from Chapter I of [31].

We indicate the set $\{1,\dots,p-1\}$ with $A$ and define the two representation functions

$$\begin{aligned}r(n) &= \#\big\{(a,b) : a,b \in A, a+b=n\big\} \\ r_-(n) &= \#\big\{(a,b) : a,b \in A, a<b, a+b=n\big\}\end{aligned}$$

so that the expression of $\dim A^p(\mathbb{C}^m[4])$ takes the form

$$\dim A^p(\mathbb{C}^m[4]) = 1 + \binom{4}{2}p + \binom{4}{3}p + \frac{1}{2}\binom{4}{2}\left[\frac{p}{2}\right] + \binom{4}{2}\sum_{n=3}^{p} r_-(n) + 2\binom{4}{2}\sum_{n=2}^{p} r(n)$$

Note that $r(0) = r_-(0) = r(1) = r_-(1) = r_-(2) = 0$ and for $n \geq 2$ we have that

$$r(n) = 2r_-(n) + \begin{cases} 1 & \text{if } n \text{ is even and } \frac{n}{2} \in A \\ 0 & \text{otherwise}\end{cases}$$

When $2 \leq n \leq p$, if $n$ is even then $n/2 \in A$ and we can write

$$\begin{aligned}r(n) &= 2r_-(n) + \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}\end{cases} \\ &= 2r_-(n) + \frac{1+(-1)^n}{2}\end{aligned}$$

For our purpose, it is more convenient to write

$$r_-(n) = \frac{1}{2}\left(r(n) - \frac{1+(-1)^n}{2}\right)$$

convenient to write Since $r(n) = n-1$, it follows that

$$r(2) + \cdots + r(p) = \frac{1}{2}p(p-1)$$

and therefore

$$r_-(3) + \cdots + r_-(p) = \frac{1}{4}\left(p(p-1) - \frac{2p-1+(-1)^p}{2}\right)$$

We can now deduce the dimension of $A^p(\mathbb{C}^m[4])$ when $p < m$

$$\begin{aligned}
\dim A^p(\mathbb{C}^m[4]) &= 1 + \binom{4}{2}p + \binom{4}{3}p + \frac{1}{2}\binom{4}{2}\left[\frac{p}{2}\right] \\
&+ \binom{4}{2}\frac{1}{4}\left(p(p-1) - \frac{2p-1+(-1)^p}{2}\right) + 2\binom{4}{2}\frac{1}{2}p(p-1) \\
&= 1 + 3\left[\frac{p}{2}\right] + \left(\binom{4}{2} + \binom{4}{3} - \frac{1}{2}\binom{4}{2} - \binom{4}{2}\right)p \\
&+ \frac{5}{4}\binom{4}{2}p^2 + \frac{1}{8}\binom{4}{2}(1-(-1)^p) \\
&= 1 + \frac{3}{4}(1-(-1)^p) + 3\left[\frac{p}{2}\right] + p + \frac{15}{2}p^2 \\
&= \begin{cases} 1 + \frac{3}{2}p + p + \frac{15}{2}p^2 & \text{if } p \text{ is even} \\ 1 + \frac{3}{2} + \frac{3}{2}(p-1) + p + \frac{15}{2}p^2 & \text{if } p \text{ is odd} \end{cases} \\
&= 1 + \frac{5}{2}(p + 3p^2)
\end{aligned}$$

$\square$

We write the values of $\dim A^p(\mathbb{C}^m[4])$, in the case $0 \leq p < m$, for several $p$'s:

| $p$ | $\dim A^p(\mathbb{C}^m[4])$ |
|---|---|
| 0 | 1 |
| 1 | 11 |
| 2 | 36 |
| 3 | 76 |
| 4 | 131 |
| 5 | 201 |
| 6 | 286 |

We now come to examine what happens when the codimension $p$ is at least $m$. As in the previous paragraph, we obtain the general rule (see (b)) after the study of some low cases.

$(a)$ $\mathbb{C}^2[4]$

We search for a basis of the Chow groups $A^p(\mathbb{C}^2[4])$ among the generators (4.10) in the case $m = 2$.

The relations (4.2$''$) are $(D_{ij} + D_{ijh} + D_{ijk} + D_{1234})^2$, therefore the unique condition on the elements (4.10) in codimension $p = 2$ allows us to eliminate the $D_{ij}^2$'s which are generated by some other elements and a basis of the 2-cocycles $A^2(\mathbb{C}^2[4])$ is given by

$$\mathscr{B}_{2,2,4} = \{D_{ijh}^2, D_{1234}^2, D_{ij} \cdot D_{hk}, D_{ij} \cdot D_{ijh}, D_{ij} \cdot D_{1234}, D_{ijh} \cdot D_{1234}\}_{i,j,h,k}$$

and we deduce that

$$\dim(A^2(\mathbb{C}^2[4])) = \binom{4}{3} + 1 + \frac{1}{2}\binom{4}{2} + 2\binom{4}{2} + \binom{4}{2} + \binom{4}{3} = 30$$

In order to find a basis of $A^3(\mathbb{C}^2[4])$ we first eliminate the products which contain the factor $D_{ij}^2$, that is $D_{ij}^2 \cdot D_{1234}, D_{ij}^3, D_{ij}^2 \cdot D_{hk}, D_{ij}^2 \cdot D_{ijh}$.

Secondly we must consider the conditions given by (4.1″) together with the relations in codimension $m = 2$ in the sense that we analyse the products

$$D_{ij} \cdot D_{hk}^2, D_{ij} \cdot D_{ijh}^2, D_{ij} \cdot D_{1234}^2, D_{ijh} \cdot D_{1234}^2, D_{ijh}^3, D_{ijh}^2 \cdot D_{1234}, D_{1234}^3$$

In the following calculations we will employ (4.2″). Applying such a relation to $D_{ijh}^2$ we obtain

$$\begin{aligned} D_{ij} \cdot D_{ijh}^2 \quad &= - D_{ij} \cdot (D_{ij}^2 + D_{ijk}^2 + D_{1234}^2 + 2D_{ij} \cdot D_{ijh} + 2D_{ij} \cdot D_{ijk} \\ &\quad + 2D_{ij} \cdot D_{1234} + 2D_{ijh} \cdot D_{1234} + 2D_{ijk} \cdot D_{1234}) \\ &= - (D_{ij}^3 + D_{ij} \cdot D_{ijk}^2 + D_{ij} \cdot D_{1234}^2 + 2D_{ij}^2 \cdot D_{ijh} + 2D_{ij}^2 \cdot D_{ijk}) \end{aligned}$$

$$\begin{aligned} D_{ij} \cdot D_{ijh}^2 \quad &= - D_{ij} \cdot (D_{ih}^2 + D_{ihk}^2 + D_{1234}^2 + 2D_{ih} \cdot D_{ijh} + 2D_{ih} \cdot D_{ihk} \\ &\quad + 2D_{ih} \cdot D_{1234} + 2D_{ijh} \cdot D_{1234} + 2D_{ihk} \cdot D_{1234}) \\ &= - (D_{ij} \cdot D_{1234}^2 + 2D_{ij} \cdot D_{ijh} \cdot D_{1234}) \end{aligned}$$

$$\begin{aligned} D_{ij} \cdot D_{ijh}^2 \quad &= - D_{ij} \cdot (D_{jh}^2 + D_{jhk}^2 + D_{1234}^2 + 2D_{jh} \cdot D_{ijh} + 2D_{jh} \cdot D_{jhk} \\ &\quad + 2D_{jh} \cdot D_{1234} + 2D_{ijh} \cdot D_{1234} + 2D_{jhk} \cdot D_{1234}) \\ &= - (D_{ij} \cdot D_{1234}^2 + 2D_{ij} \cdot D_{ijh} \cdot D_{1234}) \end{aligned}$$

When we develop $D_{ij} \cdot D_{1234}^2$ we deduce the same relations as above.

Then we claim that

$$D_{ijh} \cdot D_{1234}^2 = 0$$

To see this, it is enough to write $D_{1234}^2 = -(D_{ik}^2 + D_{ihk}^2 + D_{ijk}^2 + 2D_{ik} \cdot D_{ihk} + 2D_{ik} \cdot D_{ijk} + 2D_{ik} \cdot D_{1234} + 2D_{ihk} \cdot D_{1234} + 2D_{ijk} \cdot D_{1234})$ and we realize that all the products vanish.

The element $D_{ijh}^3$ remains because we have removed $D_{ij}^2 \cdot D_{ijh}$ and finally $D_{1234}^3$ takes part to the determination of $D_{ij}^2 \cdot D_{1234}$.

Therefore we are able to provide a basis of $A^3(\mathbb{C}^2[4])$

$$\mathscr{B}_{3,2,4} = \{D_{ijh}^3, D_{1234}^3, D_{ij} \cdot D_{1234}^2, D_{ijh}^2 \cdot D_{1234}, D_{ij} \cdot D_{hk} \cdot D_{1234}, D_{ij} \cdot D_{ijh} \cdot D_{1234}\}_{i,j,h,k}$$

and

$$\dim(A^3(\mathbb{C}^2[4])) = \binom{4}{3} + 1 + \binom{4}{2} + \binom{4}{3} + \frac{1}{2}\binom{4}{2} + 2\binom{4}{2} = 30$$

With the purpose of finding a basis of $A^4(\mathbb{C}^2[4])$ we intersect the basis of $A^3(\mathbb{C}^2[4])$ with all the (exceptional) divisors.

| basis of $A^3(\mathbb{C}^2[4])$ | $\cdot D_{ij}$ | $\cdot D_{ijh}$ | $\cdot D_{1234}$ |
|---|---|---|---|
| $D^3_{ijh}$ | $D_{ij}\cdot D^3_{ijh}$ | $D^4_{ijh}$ | $D^3_{ijh}\cdot D_{1234}$ |
| $D^3_{1234}$ | $D_{ij}\cdot D^3_{1234}$ | $D_{ijh}\cdot D^3_{1234}$ | $D^4_{1234}$ |
| $D_{ij}\cdot D^2_{1234}$ | $D^2_{ij}\cdot D^2_{1234}, D_{ij}\cdot D_{hk}\cdot D^2_{1234}$ | $D_{ij}\cdot D_{ijh}\cdot D^2_{1234}$ | $D_{ij}\cdot D^3_{1234}$ |
| $D^2_{ijh}\cdot D_{1234}$ | $D_{ij}\cdot D^2_{ijh}\cdot D_{1234}$ | $D^3_{ijh}\cdot D_{1234}$ | $D^2_{ijh}\cdot D^2_{1234}$ |
| $D_{ij}\cdot D_{hk}\cdot D_{1234}$ | $D^2_{ij}\cdot D_{hk}\cdot D_{1234}$ | 0 | $D_{ij}\cdot D_{hk}\cdot D^2_{1234}$ |
| $D_{ij}\cdot D_{ijh}\cdot D_{1234}$ | $D^2_{ij}\cdot D_{ijh}\cdot D_{1234}$ | $D_{ij}\cdot D^2_{ijh}\cdot D_{1234}$ | $D_{ij}\cdot D_{ijh}\cdot D^2_{1234}$ |

Among all these elements we can eliminate those which contain the powers $D^2_{ij}$ and those which contain the factor $D_{ijh}\cdot D^2_{1234}=0$. In this way the elements to be analysed are

$$\begin{aligned}&D^4_{ijh}, D^4_{1234},\\ &D_{ij}\cdot D^3_{1234}, D_{ij}\cdot D^3_{ijh}, D^3_{ijh}\cdot D_{1234},\\ &D_{ij}\cdot D_{hk}\cdot D^2_{1234}, D_{ij}\cdot D^2_{ijh}\cdot D_{1234}\end{aligned}$$

From the calculation we made in the determination of the previous basis, $D_{ij}\cdot D^2_{ijh}$ is obtainable by $D_{ij}\cdot D^2_{1234}$ and $D_{ij}\cdot D_{ijh}\cdot D_{1234}$, therefore $D_{ij}\cdot D^3_{ijh}$ is given by $D_{ij}\cdot D^2_{ijh}\cdot D_{1234}$ and $D_{ij}\cdot D^2_{ijh}\cdot D_{1234}$ depends on $D_{ij}\cdot D^3_{1234}$ and $D_{ij}\cdot D_{ijh}\cdot D^2_{1234}=0$.

Among the generators of the 4-cocycles there is a new vanishing given by $D_{ij}\cdot D_{hk}\cdot D^2_{1234}$. This can be seen by writing $D^2_{1234}$ starting from $D_{ih}$.

We can express $D^4_{ijh}=D^2_{ijh}\cdot D^2_{ijh}$ as

$$D^4_{ijh}=\sum_{a,b=0}^{1} c_{a,b}D^{a+b}_{ijh}\cdot(D_{ij}+D_{ijk}+D_{1234})^{2-a}\cdot(D_{ih}+D_{ihk}+D_{1234})^{2-b}$$

where $c_{a,b}=\binom{2}{a}\binom{2}{b}$. In this way we see that $D^4_{ijh}$ results as a combination of

$$\begin{aligned}&D^4_{1234}\\ &D^2_{ij}\cdot D^2_{1234},\ D^2_{ij}\cdot D_{ijh}\cdot D_{1234},\ D_{ij}\cdot D^3_{1234},\ D_{ij}\cdot D^2_{ijh}\cdot D_{1234},\\ &D^2_{ih}\cdot D^2_{1234},\ D^2_{ih}\cdot D_{ijh}\cdot D_{1234},\ D_{ih}\cdot D^3_{1234},\ D_{ih}\cdot D^2_{ijh}\cdot D_{1234}\end{aligned}$$

(the other terms vanish). From (4.2″) and $D_{ijh}\cdot D^2_{1234}=0$, we infer that $D^2_{ij}\cdot D^2_{1234}$ is given by $D^4_{1234}$ and $D_{ij}\cdot D^3_{1234}$. In addition $D^2_{ij}\cdot D_{ijh}\cdot D_{1234}$ is defined by $D^3_{ijh}\cdot D_{1234}$ and $D_{ij}\cdot D^3_{1234}$.

Finally, we find that $D^4_{ijh}$ is a combination of $D_{ij}\cdot D^3_{1234}$, $D_{ih}\cdot D^3_{1234}$, $D^3_{ijh}\cdot D_{1234}$ and $D^4_{1234}$ which are independent. Indeed, expanding $D^3_{ijh}\cdot D_{1234}$ we get $2D_{ij}\cdot D^3_{1234}-D^2_{ij}\cdot D_{ijh}\cdot D_{1234}$, which is the relation we have considered to eliminate the corresponding term $D^2_{ij}\cdot D_{ijh}\cdot D_{1234}$. The same holds for $D_{ij}\cdot D^3_{1234}$ with respect to $D_{ij}\cdot D^2_{ijh}\cdot D_{1234}$ and for $D^4_{1234}$ with respect to $D^2_{ij}\cdot D^2_{1234}$.

Therefore a basis of $A^4(\mathbb{C}^2[4])$ is

$$\mathscr{B}_{4,2,4}=\{D^4_{1234}, D_{ij}\cdot D^3_{1234}, D^3_{ijh}\cdot D_{1234}\}_{i,j,h,k}$$

and

$$\dim(A^4(\mathbb{C}^2[4])) = 1 + \binom{4}{2} + \binom{4}{3} = 11$$

We deduce a basis of $A^5(\mathbb{C}^2[4])$ studying the products generated by the basis of $A^4(\mathbb{C}^2[4])$

| basis of $A^4(\mathbb{C}^2[4])$ | $\cdot D_{ij}$ | $\cdot D_{ijh}$ | $\cdot D_{1234}$ |
|---|---|---|---|
| $D^4_{1234}$ | $D_{ij} \cdot D^4_{1234}$ | $D_{ijh} \cdot D^4_{1234}$ | $D^5_{1234}$ |
| $D_{ij} \cdot D^3_{1234}$ | $D^2_{ij} \cdot D^3_{1234}, D_{ij} \cdot D_{hk} \cdot D^3_{1234}$ | $D_{ij} \cdot D_{ijh} \cdot D^3_{1234}$ | $D_{ij} \cdot D^4_{1234}$ |
| $D^3_{ijh} \cdot D_{1234}$ | $D_{ij} \cdot D^3_{ijh} \cdot D_{1234}$ | $D^4_{ijh} \cdot D_{1234}$ | $D^3_{ijh} \cdot D^2_{1234}$ |

We can immediately state that $D_{ij} \cdot D_{hk} \cdot D^3_{1234} = 0$, $D_{ijh} \cdot D^4_{1234} = 0$, $D_{ij} \cdot D_{ijh} \cdot D^3_{1234} = 0$ and $D^3_{ijh} \cdot D^2_{1234} = 0$.

Since $D^4_{ijh}$ is defined by $D_{ij} \cdot D^3_{1234}$ and $D^4_{1234}$, we can eliminate $D^4_{ijh} \cdot D_{1234}$.

Then $D_{ij} \cdot D^4_{1234} = 0$ because $D_{ij} \cdot D^2_{1234}$ derives from $D_{ij} \cdot D^2_{ijh}$ and $D_{ij} \cdot D_{ijh} \cdot D_{1234}$ and we know that $D_{ijh} \cdot D^2_{1234} = 0$.

$D_{ij} \cdot D^3_{ijh} \cdot D_{1234} = 0$, indeed from a previous remark $D_{ij} \cdot D^2_{ijh} \cdot D_{1234}$ is determined by $D_{ij} \cdot D^3_{1234}$.

The term $D^2_{ij} \cdot D^3_{1234}$ contains the factor $D^2_{ij}$ and therefore it can be omitted.

A basis of $A^5(\mathbb{C}^2[4])$ is therefore generated by the unique element

$$\mathscr{B}_{5,2,4} = \{D^5_{1234}\}$$

The last group to be analized is $A^6(\mathbb{C}^2[4])$, indeed $D_{ij} \cdot D^4_{1234} \cdot D_{1234} = 0$, $D_{ijh} \cdot D^2_{1234} \cdot D^3_{1234} = 0$ and

$$\begin{aligned}
&D^6_{1234} \\
&= D^2_{1234} \cdot D^2_{1234} \cdot D^2_{1234} \\
&= \sum_{a,b,c=0}^{1} c_{abc} D^{a+b+c}_{1234} \cdot (D_{ij} + D_{ijh} + D_{ijk})^{2-a} \cdot (D_{ih} + D_{ijh} + D_{ihk})^{2-b} \cdot (D_{hk} + D_{ihk} + D_{jhk})^{2-c} \\
&= 0
\end{aligned}$$

where $c_{abc} = \binom{2}{a}\binom{2}{b}\binom{2}{c}$. Therefore $A^6(\mathbb{C}^2[4]) = 0$.

$A^{5-2p}(\mathbb{C}^2[4])$, $(b)$ $\mathbb{C}^m[4]$

We shall now try to generalize the results we found for $m = 2$ to each value of $m$.

The following study is part of the proof of Proposition 4.3.2.

The only $p$-cocycles where the relations occure are for $m \leq p$ and we will begin from $A^m$. As observed in the preceding example, in codimension $m$ the elements $D^m_{ij}$ are redundant and a basis of $A^m(\mathbb{C}^m[4])$ is given by

$$\mathscr{B}_{m,m,4} = \left\{ \begin{array}{ll} D^a_{ij} \cdot D^b_{hk} \cdot D^{m-a-b}_{1234}, & 0 \leq a, b \leq m-1, 1 \leq a+b \leq m; \\ D^a_{ij} \cdot D^b_{ijh} \cdot D^{m-a-b}_{1234}, & 0 \leq a \leq m-1, 0 \leq b \leq m, 1 \leq a+b \leq m; \\ D^m_{1234} & \end{array} \right\}_{i,j,h,k} \tag{4.12}$$

Also when $m > 2$ the generators which depend on some others are of the same type as for $m = 2$. Indeed, in order to find the dependent elements, we must look at the products which contain some $D_S^m$:

$D_{ij}^m$ can be substituted by elements like $D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{m-x-y}$, $0 \leq x \leq m-1, 0 \leq y, x+y \leq m$.

$D_{ijh}^m$ is included in $D_{ij}^m$.

Since $D_{ijh}$ appears in two relations, $D_{ij} \cdot D_{ijh}^m$ depends on $D_{ij} \cdot D_{ijh}^x \cdot D_{1234}^{m-x}$, $0 \leq x \leq m-1$.

Both $D_{ijh}^{m+1}$ and $D_{ijh}^m \cdot D_{1234}$ cannot give new conditions. $D_{ij}^m \cdot D_{ijh}^m$ has already been considered.

$D_{ijh}^{2m}$ is determined by

$$\begin{array}{lr} D_{ij}^x \cdot D_{ijh}^{a+b} \cdot D_{1234}^{2m-a-b-x} & 0 \leq a, b \leq m-1,\ 0 \leq x \leq m-a \\ D_{ih}^x \cdot D_{ijh}^{a+b} \cdot D_{1234}^{2m-a-b-x} & 0 \leq a, b \leq m-1,\ 0 \leq x \leq m-a \\ D_{ih}^x \cdot D_{ijh}^y \cdot D_{1234}^{2m-x-y} & 0 \leq x, y, x+y \leq m \end{array}$$

The first one reduces to

$$\begin{array}{lr} D_{ij}^x \cdot D_{ijh}^{y+b} \cdot D_{1234}^{2m-b-x-y} & 1 \leq b \leq m-1,\ 0 \leq x \leq m-1,\ 0 \leq y \leq m-x \\ D_{ij}^x \cdot D_{ijh}^b \cdot D_{1234}^{2m-b-x} & 1 \leq b \leq m-1,\ 0 \leq x \leq m-1 \\ D_{ij} \cdot D_{1234}^{2m-1} & \end{array}$$

whereas the third one to

$$D_{ih}^x \cdot D_{1234}^{2m-x} \qquad 0 \leq x \leq m-1$$

In a while we will see that $D_{ijh}^m \cdot D_{1234}^m = 0$.

$D_{1234}^m$ is present in $D_{ij}^m$.

$D_{ij} \cdot D_{1234}^m$ takes part to determine $D_{ij} \cdot D_{ijh}^m$ on one side and $D_{ij} \cdot D_{hk}^m$ on th other one.

$D_{ijh} \cdot D_{1234}^m = 0$ because $D_{1234}^m$ can be seen as sum of terms like $D_{ik}^x \cdot D_{ihk}^y \cdot D_{1234}^{m-x-y}$, $0 \leq x, y \leq m, 1 \leq x+y \leq m$.

$D_{ij} \cdot D_{hk} \cdot D_{1234}^m = 0$, indeed it is sufficient to express $D_{1234}^m$ starting from $D_{ih}$.

$D_{ij} \cdot D_{ijh} \cdot D_{1234}^m$, $D_{ij} \cdot D_{hk}^m \cdot D_{1234}^m$ $D_{ij} \cdot D_{ijh}^m \cdot D_{1234}^m$, $D_{ij}^m \cdot D_{ijh} \cdot D_{1234}^m$ have already been studied.

$D_{ij} \cdot D_{1234}^{2m} = 0$, for example because $D_{ij} \cdot D_{1234}^m$ can be seen as a sum of $D_{ij} \cdot D_{ijh}^{m-a} \cdot D_{1234}^a$, $0 \leq a \leq m-1$, and $D_{ijh} \cdot D_{1234}^m = 0$.

The last element to be examined is $D_{1234}^{3m}$. Exploiting $D_{1234}^m$ once starting from $D_{ij}$, then from $D_{ih}$ and finally from $D_{hk}$, it results that $D_{1234}^{3m} = 0$.

We can summerize this analysis as follows:

1. if $p \geq m$ we eliminate the terms which contain the factors $D_{ij}^m$

2. if $p \geq m+1$ we eliminate the terms which contain the factors $D_{ij} \cdot D_{ijh}^m, D_{ijh} \cdot D_{1234}^m = 0$
3. if $p \geq m+2$ we eliminate the terms which contain the factors $D_{ij} \cdot D_{hk} \cdot D_{1234}^m = 0$
4. if $p \geq 2m$ we eliminate the terms which contain the factors $D_{ijh}^{2m}$
5. if $p \geq 2m+1$ we eliminate the terms which contain the factors $D_{ij} \cdot D_{1234}^{2m} = 0$
6. if $p \geq 3m$ we eliminate the terms which contain the factor $D_{1234}^{3m} = 0$

In order to see if isomorphisms hold among the Chow groups we first determine the dimensions of such groups. In this view we count the number of dependent elements.

1. When $p \geq m$, the terms which contain $D_{ij}^m$ are the elements given by the following expressions

$$D_{ij}^m \cdot \begin{cases} D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^{p-m-a-b} & 0 \leq a, b, a+b \leq p-m \\ D_{ij}^a \cdot D_{hk}^b \cdot D_{1234}^{p-m-a-b} & 0 \leq a, b, a+b \leq p-m \end{cases}$$

which we can rewrite in the form

$$\{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : m \leq x \leq p,\ 0 \leq y \leq p-m,\ m \leq x+y \leq p\}_{i,j,h} \tag{4.13}$$

$$\{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : m \leq x \leq p,\ 0 \leq z \leq p-m,\ m \leq x+z \leq p\}_{i,j,h,k} \tag{4.14}$$

Now we want to count such elements.

As regards those of type (4.13), for each fixed $x$ we can choose $y$ in $\{m-x, \ldots, p-x\}$ under the condition $0 \leq y \leq p-m$, that is $0 \leq y \leq p-x$. In calculating the number of type (4.13) elements, we separate the cases $y = 0$ and $y \neq 0$ and obtain

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : m \leq x \leq p, 0 \leq y \leq p-m, m \leq x+y \leq p\}_{i,j,h} \\
&= \binom{4}{2}(p-(m-1)) + 2\binom{4}{2}\sum_{x=m}^{p-1}(p-x) \\
&= \binom{4}{2}(p-m+1) + 2\binom{4}{2}\Big(p(p-1-(m-1)) - m(p-m) - \frac{(p-m-1)(p-m)}{2}\Big) \\
&= \binom{4}{2}(p-m+1) + 2\binom{4}{2}(p-m)\Big(p-m-\frac{p-m-1}{2}\Big) \\
&= \binom{4}{2}(p-m+1)^2
\end{aligned} \tag{4.13$'$}$$

Concerning the products of type (4.14), for each fixed $x$ we can choose $z$ in $\{m-x, \ldots, p-x\}$ under the condition $0 \leq z \leq p-m$, that is $0 \leq z \leq p-x$. When $p \geq 2m$ we shall distinguish the terms with $z < m$ from $z \geq m$ which are repeated.

If $p < 2m$ then $p-m < m \leq x$, that is $p-x < m$ and so $z < m$.

If $p \geq 2m$ then $m \leq p-m$ and therefore when $m \leq x \leq p-m$ we have that $m \leq p-x$ whereas when $p-m+1 \leq x \leq p$ it results that $p-x+1 \leq m$ and we deduce the cardinality of the set (4.14)

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : m \leq x \leq p, 0 \leq z \leq p-m, m \leq x+z \leq p\}_{i,j,h,k} \\
&= \begin{cases} \binom{4}{2} \sum_{x=m}^{p} (p-x+1) & p < 2m \\ \underbrace{\binom{4}{2}\Big(\sum_{x=m}^{p-m+1} m + \sum_{x=p-m+2}^{p} (p-x+1)\Big)}_{z<m} + \underbrace{\frac{1}{2}\binom{4}{2}\sum_{x=m}^{p-m}(p-x-m+1)}_{z\geq m} & p \geq 2m \end{cases} \\
&= \begin{cases} \frac{3}{2}(p-m+1)(p-m+2) & p < 2m \\ \frac{3}{2}(p-2m+2)(p+2m+1) + 3m(m-1) & p \geq 2m \end{cases}
\end{aligned} \tag{4.14$'$}$$

2. When $p \geq m+1$, we must subtract the elements which contain $D_{ij} \cdot D_{ijh}^m$ and $D_{ijh} \cdot D_{1234}^m$

$$\begin{aligned}
&\{D_{ij} \cdot D_{ijh}^m \cdot D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^{p-(m+1)-a-b} : 0 \leq a, b, a+b \leq p-(m+1)\} \\
&= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\{D_{ijh} \cdot D_{1234}^m \cdot D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^{p-(m+1)-a-b} : 0 \leq a, b, a+b \leq p-(m+1)\} \\
&= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}
\end{aligned} \tag{4.16}$$

In (4.15), for each fixed $x$ in $\{1, \ldots, p-m\}$ we can choose $y$ in $\{m+1-x, \ldots, p-x\}$. Since $m \leq y \leq p-1$, for each $x$ there are $p-x-(m-1)$ possibilities for $y$

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}_{i,j,h} \\
&= 2\binom{4}{2}\sum_{x=1}^{p-m}(p-x-(m-1)) = 2\binom{4}{2}\left((p-m)(p-m+1) - \frac{(p-m)(p-m+1)}{2}\right) \\
&= \binom{4}{2}(p-m)(p-m+1)
\end{aligned} \tag{4.15$'$}$$

In (4.16), for each fixed $y$ in $\{1, \ldots, p-m\}$ we can choose $x$ in $\{1-y, \ldots, p-m-y\}$. Since $0 \leq x \leq p-(m+1)$, for each $y$ there are $p-y-m+1$ possibilities for $y$. We distinguish the cases $x=0$ and $x \neq 0$

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}_{i,j,h} \\
&= \underbrace{\binom{4}{3}(p-m)}_{x=0} + \underbrace{3\binom{4}{3}\sum_{y=1}^{p-m-1}(p-m-y)}_{x\neq 0} \\
&= \binom{4}{3}(p-m) + 3\binom{4}{3}\Big((p-m)(p-m-1) - \frac{(p-m)(p-m-1)}{2}\Big) \\
&= \binom{4}{3}(p-m)\Big(1+\frac{3}{2}(p-m-1)\Big) = 4(p-m)+6(p-m)(p-m-1) \qquad (4.16')
\end{aligned}$$

3. When $p \geq m+2$, the new redundant products are those containing $D_{ij} \cdot D_{hk} \cdot D_{1234}^m$

$$\begin{aligned}
&\{D_{ij} \cdot D_{hk} \cdot D_{1234}^m \cdot D_{ij}^a \cdot D_{hk}^b \cdot D_{1234}^{p-(m+2)-a-b} : 0 \leq a, b, a+b \leq p-(m+2)\} \\
&= \{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : 1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\} \qquad (4.17)
\end{aligned}$$

Every time we fix $x$, $z$ satisfies $2-x \leq z \leq p-m-x$ and $1 \leq z \leq p-(m+1)$ that is $1 \leq z \leq p-m-x$. Since the products above are symmetric with respect to the pairs $\{i,j\}$ and $\{h,k\}$ we find that

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : 1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\}_{i,j,h,k} \\
&= \frac{1}{2}\binom{4}{2}\sum_{x=1}^{p-m-1}(p-m-x) \\
&= 3\Big((p-m)(p-m-1) - \frac{(p-m)(p-m-1)}{2}\Big) \\
&= \frac{3}{2}(p-m)(p-m-1) \qquad (4.17')
\end{aligned}$$

4. When $p \geq 2m$, we must consider the elements containing $D_{ijh}^{2m}$ which are

$$\begin{aligned}
&\{D_{ijh}^{2m} \cdot D_{ij}^a \cdot D_{ijh}^b \cdot D_{1234}^{p-2m-a-b} : 0 \leq a, b, a+b \leq p-2m\} \\
&= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\} \qquad (4.18)
\end{aligned}$$

Applying the same argument as in the previous cases, for each fixed $y$ the value of $x$ varies in $\{2m-y, \dots, p-y\} \cap \{0, \dots, p-2m\}$, that is $x \in \{0, \dots, p-y\}$. Hence

$$\begin{aligned}
&\#\{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\}_{i,j,h} \\
&= \underbrace{\binom{4}{3}(p-2m+1)}_{x=0} + \underbrace{3\binom{4}{3}\sum_{y=2m}^{p-1}(p-y)}_{x\neq 0} \\
&= 4(p-2m+1) + 12\Big(p(p-2m) - 2m(p-2m) - \frac{(p-2m-1)(p-2m)}{2}\Big) \\
&= 4(p-2m+1) + 6(p-2m+1)(p-2m) \qquad (4.18')
\end{aligned}$$

5. When $p \geq 2m+1$, there are new other conditions given by $D_{ij} \cdot D_{1234}^{2m}$

$$\begin{aligned}&\{D_{ij} \cdot D_{1234}^{2m} \cdot D_{ij}^{a} \cdot D_{ijh}^{b} \cdot D_{1234}^{p-(2m+1)-a-b} : 0 \leq a, b, a+b \leq p-(2m+1)\}\\ &= \{D_{ij}^{x} \cdot D_{ijh}^{y} \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&\{D_{ij} \cdot D_{1234}^{2m} \cdot D_{ij}^{a} \cdot D_{hk}^{b} \cdot D_{1234}^{p-(2m+1)-a-b} : 0 \leq a, b, a+b \leq p-(2m+1)\}\\ &= \{D_{ij}^{x} \cdot D_{hk}^{z} \cdot D_{1234}^{p-x-z} : 1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\}\end{aligned} \tag{4.20}$$

In (4.19), for each $x$ we can choose $y$ in $\{0, \ldots, p-2m-x\}$ and the number of such elements is

$$\begin{aligned}&\#\{D_{ij}^{x} \cdot D_{ijh}^{y} \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\}_{i,j,h}\\ &= \underbrace{\binom{4}{2}(p-2m)}_{y=0} + \underbrace{2\binom{4}{2}\sum_{x=1}^{p-2m-1}(p-2m-x)}_{y\neq 0}\\ &= 6(p-2m) + 12\Big((p-2m)(p-2m-1) - \frac{(p-2m-1)(p-2m)}{2}\Big)\\ &= 6(p-2m)^2\end{aligned} \tag{4.19$'$}$$

In an analogous way, we can state that the number of type (4.20) elements is

$$\begin{aligned}&\#\{D_{ij}^{x} \cdot D_{hk}^{z} \cdot D_{1234}^{p-x-z} : 1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\}_{i,j,h,k}\\ &= \underbrace{\binom{4}{2}(p-2m)}_{z=0} + \underbrace{\frac{1}{2}\binom{4}{2}\sum_{x=1}^{p-2m-1}(p-2m-x)}_{z\neq 0}\\ &= 6(p-2m) + \frac{3}{2}(p-2m)(p-2m-1)\end{aligned} \tag{4.20$'$}$$

6. The case $p \geq 3m$ will be treated in Proposition 4.3.4.

Now we are going to determine $\dim A^p(\mathbb{C}^m[4])$, $\forall\ p \geq m$. We will deal with the cases $p < 2m$ and $p \geq 2m$ separately because they are subject to different laws.

**Notation 4.3.1.** In calculating the dimension of $p$-cocycles for $p \geq m$ it will be useful to denote by $d_p(\mathbb{C}^m[4])$ the number of elements of the set (4.10) which corresponds to the dimension of the group $A^p(\mathbb{C}^m[4])$ when $p < m$, that is $d_p(\mathbb{C}^m[4]) = 1 + \frac{5}{2}p(1+3p)$, $\forall p = 0, \ldots, 3m$. If $p < m$ then $\dim A^p(\mathbb{C}^m[4]) = d_p(\mathbb{C}^m[4])$, otherwise $\dim A^p(\mathbb{C}^m[4]) < d_p(\mathbb{C}^m[4])$.

**Proposition 4.3.2.** *For each $m \le p \le 2m-1$*

$$\dim(A^p(\mathbb{C}^m[4])) = \frac{5}{2}p(1+3p) - \frac{5}{2}(p-m)(7+9(p-m)) - 5 \tag{4.21}$$

*Proof.* For each $p \ge m$, $\dim(A^p(\mathbb{C}^m[4]))$ is obtained by subtracting the number of the redundant elements from $d_p(\mathbb{C}^m[4]))$.

We must take care of repeated products which appear in more than one set among those determined by (4.13),(4.14),(4.15),(4.16) and (4.17). For this reason we are going to count such elements which occur at the intersection of each pair of the mentioned sets. To this aim we have decided to write all the products in the form $D_{ij}^x \cdot D_{hk}^z \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y-z}$.

By a simple calculation which we prefer to omit, we find that the only repeated elements are those given by

$$D_{ij}^x \cdot D_{1234}^{p-x} \qquad\qquad m \le x \le p$$

They belong both to the set (4.13) and to the set (4.14) therefore their number will be added once since we have eliminated it twice.

From (4.13′)–(4.17′) we infer that

$$\begin{aligned}
\dim(A^p(\mathbb{C}^m[4])) \quad &= d_p(\mathbb{C}^m[4]) \\
&\quad - \binom{4}{2}(p-m+1)^2 - \frac{1}{2}\binom{4}{2}(p-m+1)(p-m+2) \\
&\quad - \binom{4}{2}(p-m)(p-m+1) - 4(p-m) - 6(p-m)(p-m-1) \\
&\quad - \frac{3}{2}(p-m)(p-m-1) \\
&\quad + \binom{4}{2}(p-m+1) \\
&= d_p(\mathbb{C}^m[4]) \\
&\quad - 3(p-m+1)\big(2(p-m+1) + (p-m+2) + 2(p-m) - 2\big) \\
&\quad - (p-m)\big(4 + 6(p-m-1) + \frac{3}{2}(p-m-1)\big) \\
&= 1 + \frac{5}{2}p(1+3p) - 3(p-m+1)(5(p-m)+2) \\
&\quad - (p-m)\Big(\frac{15}{2}(p-m) - \frac{7}{2}\Big) \\
&= 1 + \frac{5}{2}p(1+3p) - \frac{45}{2}(p-m)^2 - \frac{35}{2}(p-m) - 6 \\
&= \frac{5}{2}p(1+3p) - \frac{5}{2}(p-m)\big(7+9(p-m)\big) - 5
\end{aligned}$$

□

In the following lemma, we determine the number of redundant elements which are present in various sets among those of type (4.13)–(4.20) with the aim of finding the dimension of $A^p(\mathbb{C}^m[4])$ when $2m \le p \le 3m-1$.

**Lemma 4.3.1.** *For all $2m \le p \le 3m-1$, the number of elements which belong to more than one set of redundant elements is*

$$18 + 6(p-m) + \frac{75}{2}(p-2m) + \frac{69}{2}(p-2m)^2 \tag{4.22}$$

*Proof.* In the sequel we will make an abuse of notation by writing $(\alpha) \cap (\beta)$ in place of $A \cap B$ when $A$ and $B$ are sets which are referred to the numbers $(\alpha)$ and $(\beta)$ respectively.

- $(4.13) \cap (4.14)$: Since

$$\begin{aligned}&\{m \le x \le p, 0 \le y \le p-m, m \le x+y \le p\}\\ &\cap \{m \le x \le p, 0 \le z \le p-m, m \le x+z \le p\}\\ &= \{m \le x \le p, y = z = 0\}\end{aligned}$$

it follows that

$$\begin{aligned}&(4.13) \cap (4.14) = \{D_{ij}^x \cdot D_{1234}^{p-x} : m \le x \le p\}_{i,j}\\ &\#((4.13) \cap (4.14)) = 6(p-m+1)\end{aligned} \tag{4.23}$$

- $(4.13) \cap (4.15)$: In this case, since $p - m \ge m$,

$$\begin{aligned}&\{m \le x \le p, 0 \le y \le p-m, m \le x+y \le p\}\\ &\cap \{1 \le x \le p-m, m \le y \le p-1, m+1 \le x+y \le p\}\\ &= \{m \le x \le p-m, m \le y \le p-m, m+1 \le x+y \le p\}\end{aligned}$$

and

$$(4.13) \cap (4.15) = \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : m \le x \le p-m, m \le y \le p-m, m+1 \le x+y \le p\}_{i,j,h}$$

$$\begin{aligned}\#((4.13) \cap (4.15)) &= 2\binom{4}{2}\sum_{x=m}^{p-m}(p-x-m+1)\\ &= 12(p-m+1)(p-2m+1) - 12(m(p-2m+1) + \frac{(p-2m)(p-2m+1)}{2})\\ &= 6(p-2m+1)(p-2m+2)\end{aligned} \tag{4.24}$$

- $(4.13) \cap (4.16)$:

$$\begin{aligned}&\{m \le x \le p, 0 \le y \le p-m, m \le x+y \le p\}\\ &\cap \{0 \le x \le p-(m+1), 1 \le y \le p-m, 1 \le x+y \le p-m\}\\ &= \begin{cases}\emptyset & if p = 2m\\ \{m \le x \le p-m-1, 1 \le y \le p-m, m \le x+y \le p-m\} & if p \ge 2m+1\end{cases}\end{aligned}$$

therefore

$$
\begin{aligned}
&(4.13)\cap(4.16)\\
&=\begin{cases}\emptyset & if p=2m\\ \{D^x_{ij}\cdot D^y_{ijh}\cdot D^{p-x-y}_{1234} : m\le x\le p-m-1, 1\le y\le p-m-x\}_{i,j,h} & if p\ge 2m+1\end{cases}\\
&\#((4.13)\cap(4.16))=\begin{cases}0 & if p=2m\\ 12\sum_{x=m}^{p-m-1}(p-m-x) & if p\ge 2m+1\end{cases}\\
&=6(p-2m)(p-2m+1)
\end{aligned}
\tag{4.25}
$$

- $(4.13)\cap(4.17)$:

$$
\begin{aligned}
&\{m\le x\le p, 0\le y\le p-m, m\le x+y\le p\}\\
&\cap\{1\le x, z\le p-(m+1), 2\le x+z\le p-m\}=\emptyset
\end{aligned}
$$

- $(4.13)\cap(4.18)$: Since we are considering $2m\le p\le 3m-1$ it follows that $p-2m<m$ and hence

$$
\begin{aligned}
&\{m\le x\le p, 0\le y\le p-m, m\le x+y\le p\}\\
&\cap\{0\le x\le p-2m, 2m\le y\le p, 2m\le x+y\le p\}=\emptyset
\end{aligned}
$$

- $(4.13)\cap(4.19)$ and $(4.13)\cap(4.20)$: In an analogous way to the previous case

$$
\begin{aligned}
&\{m\le x\le p, 0\le y\le p-m, m\le x+y\le p\}\\
&\cap\{1\le x\le p-2m, 0\le y\le p-2m-1, 1\le x+y\le p-2m\}=\emptyset
\end{aligned}
$$

and

$$
\begin{aligned}
&\{m\le x\le p, 0\le y\le p-m, m\le x+y\le p\}\\
&\cap\{1\le x\le p-2m, 0\le z\le p-2m-1, 1\le x+z\le p-2m\}=\emptyset
\end{aligned}
$$

- $(4.14)\cap(4.15)$:

$$
\begin{aligned}
&\{m\le x\le p, 0\le z\le p-m, m\le x+z\le p\}\\
&\cap\{1\le x\le p-m, m\le y\le p-1, m+1\le x+y\le p\}=\emptyset
\end{aligned}
$$

- $(4.14)\cap(4.16)$:

$$
\begin{aligned}
&\{m\le x\le p, 0\le z\le p-m, m\le x+z\le p\}\\
&\cap\{0\le x\le p-(m+1), 1\le y\le p-m, 1\le x+y\le p-m\}=\emptyset
\end{aligned}
$$

- $(4.14)\cap(4.17)$: Also in this case we use the fact $p<3m$ because the values of $z$ are not repeated in $x$ and we get

$$
\begin{aligned}
&\{m\le x\le p, 0\le z\le p-m, m\le x+z\le p\}\\
&\cap\{1\le x, z\le p-(m+1), 2\le x+z\le p-m\}\\
&=\begin{cases}\emptyset & if p=2m\\ \{m\le x\le p-m-1, 1\le z\le p-m-1, m\le x+z\le p-m\} & if p\ge 2m+1\end{cases}
\end{aligned}
$$

$$\begin{aligned}&(4.14) \cap (4.17)\\ &= \begin{cases} \emptyset & if p = 2m \\ \{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : m \leq x \leq p-m-1, 1 \leq z \leq p-m-x\}_{i,j,h,k} & if p \geq 2m+1 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\#((4.14) \cap (4.17)) &= \begin{cases} 0 & if p = 2m \\ 6\sum_{x=m}^{p-m-1}(p-m-x) & if p \geq 2m+1 \end{cases}\\ &= 3(p-2m)(p-2m+1)\end{aligned} \tag{4.26}$$

- $(4.14) \cap (4.18)$, $(4.14) \cap (4.19)$ and $(4.14) \cap (4.20)$: Being $p - 2m < m$

$$\begin{aligned}&\{m \leq x \leq p, 0 \leq z \leq p-m, m \leq x+z \leq p\}\\ &\cap \{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\} = \emptyset\end{aligned}$$

$$\begin{aligned}&\{m \leq x \leq p, 0 \leq z \leq p-m, m \leq x+z \leq p\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\} = \emptyset\end{aligned}$$

and

$$\begin{aligned}&\{m \leq x \leq p, 0 \leq z \leq p-m, m \leq x+z \leq p\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\} = \emptyset\end{aligned}$$

- $(4.15) \cap (4.16)$:

$$\begin{aligned}&\{1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}\\ &\cap \{0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}\\ &= \begin{cases} \emptyset & if p = 2m \\ \{1 \leq x \leq p-m-1, m \leq y \leq p-m, m+1 \leq x+y \leq p-m\} & if p \geq 2m+1 \end{cases}\end{aligned}$$

hence we state that if $p \geq 2m+1$

$$\begin{aligned}&(4.15) \cap (4.16)\\ &= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-m-1, m \leq y \leq p-m-x\}_{i,j,h}\\ &= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m, m \leq y \leq p-m-x\}_{i,j,h}\end{aligned}$$

and

$$\begin{aligned}\#((4.15) \cap (4.16)) &= \begin{cases} 0 & if p = 2m \\ 12\sum_{x=1}^{p-2m}(p-m-x-m+1) & if p \geq 2m+1 \end{cases}\\ &= 6(p-2m)(p-2m+1)\end{aligned} \tag{4.27}$$

- (4.15) $\cap$ (4.17):

$$\{1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}$$
$$\cap \{1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\} = \emptyset$$

- (4.15) $\cap$ (4.18):

$$\begin{aligned} &\{1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\} \\ &\cap \{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\} \\ &= \begin{cases} \emptyset & if p = 2m \\ \{1 \leq x \leq p-2m, 2m \leq y \leq p-1, 2m \leq x+y \leq p\} & if p \geq 2m+1 \end{cases} \end{aligned}$$

therefore

$$\begin{aligned} &(4.15) \cap (4.18) \\ &= \begin{cases} \emptyset & if p = 2m \\ \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m, 2m \leq y \leq p-x\} & if p \geq 2m+1 \end{cases} \\ &\#((4.15) \cap (4.18)) \\ &= \begin{cases} 0 & if p = 2m \\ 12 \sum_{x=1}^{p-2m} (p-x-2m+1) & if p \geq 2m+1 \end{cases} \\ &= 6(p-2m)(p-2m+1) \end{aligned} \tag{4.28}$$

- (4.15) $\cap$ (4.19): Because $p < 3m$

$$\{1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}$$
$$\cap \{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\} = \emptyset$$

- (4.15) $\cap$ (4.20):

$$\{1 \leq x \leq p-m, m \leq y \leq p-1, m+1 \leq x+y \leq p\}$$
$$\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\} = \emptyset$$

- (4.16) $\cap$ (4.17):

$$\{0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}$$
$$\cap \{1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\} = \emptyset$$

- (4.16) $\cap$ (4.18): $p-m < 2m$ therefore

$$\{0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}$$
$$\cap \{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\} = \emptyset$$

- (4.16) $\cap$ (4.19):

$$\begin{aligned}&\{0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\}\\ &= \begin{cases} \emptyset & if p = 2m, 2m+1 \\ \{1 \leq x \leq p-2m, 1 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\} & if p \geq 2m+2 \end{cases}\end{aligned}$$

therefore when $p \geq 2m+2$

$$\begin{aligned}&(4.16) \cap (4.19)\\ &= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m, 1 \leq y \leq p-2m-x\}_{i,j,h}\\ &= \{D_{ij}^x \cdot D_{ijh}^y \cdot D_{1234}^{p-x-y} : 1 \leq x \leq p-2m-1, 1 \leq y \leq p-2m-x\}_{i,j,h}\end{aligned}$$

and

$$\#((4.16) \cap (4.19)) = 6(p-2m)(p-2m-1) \tag{4.29}$$

- (4.16) $\cap$ (4.20):

$$\begin{aligned}&\{0 \leq x \leq p-(m+1), 1 \leq y \leq p-m, 1 \leq x+y \leq p-m\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\} = \emptyset\end{aligned}$$

- (4.17) $\cap$ (4.18):

$$\begin{aligned}&\{1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\}\\ &\cap \{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\} = \emptyset\end{aligned}$$

- (4.17) $\cap$ (4.19):

$$\begin{aligned}&\{1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\} = \emptyset\end{aligned}$$

- (4.17) $\cap$ (4.20):

$$\begin{aligned}&\{1 \leq x, z \leq p-(m+1), 2 \leq x+z \leq p-m\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\}\\ &= \begin{cases} \emptyset & if p = 2m, 2m+1 \\ \{1 \leq x \leq p-2m, 1 \leq z \leq p-2m-1, 2 \leq x+z \leq p-2m\} & if p \geq 2m+2 \end{cases}\end{aligned}$$

and it follows that if $p \geq 2m+2$ then

$$\begin{aligned}&(4.17) \cap (4.20)\\ &= \{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : 1 \leq x \leq p-2m, 1 \leq z \leq p-2m-x\}_{i,j,h,k}\\ &= \{D_{ij}^x \cdot D_{hk}^z \cdot D_{1234}^{p-x-z} : 1 \leq x \leq p-2m-1, 1 \leq z \leq p-2m-x\}_{i,j,h,k}\end{aligned}$$

and

$$\#((4.17) \cap (4.20)) = \frac{3}{2}(p-2m)(p-2m-1) \tag{4.30}$$

- $(4.18) \cap (4.19)$: Due to the fact that $p < 4m$, it results that

$$\begin{aligned}&\{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\} = \emptyset\end{aligned}$$

- $(4.18) \cap (4.20)$:

$$\begin{aligned}&\{0 \leq x \leq p-2m, 2m \leq y \leq p, 2m \leq x+y \leq p\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\} = \emptyset\end{aligned}$$

- $(4.19) \cap (4.20)$:

$$\begin{aligned}&\{1 \leq x \leq p-2m, 0 \leq y \leq p-2m-1, 1 \leq x+y \leq p-2m\}\\ &\cap \{1 \leq x \leq p-2m, 0 \leq z \leq p-2m-1, 1 \leq x+z \leq p-2m\}\\ &= \{1 \leq x \leq p-2m, y = z = 0\}\end{aligned}$$

and it follows that

$$(4.19) \cap (4.20) = \begin{cases} \emptyset & ifp = 2m \\ \{D_{ij}^{x} \cdot D_{1234}^{p-x} : 1 \leq x \leq p-2m\}_{i,j} & ifp \geq 2m+1 \end{cases}$$

$$\#((4.19) \cap (4.20)) = 6(p-2m) \tag{4.31}$$

From above, when $p \geq 2m+2$ the intersections among the sets of the redundant elements are expressed by the sets

$(4.13) \cap (4.14)$, $(4.13) \cap (4.15)$, $(4.13) \cap (4.16)$, $(4.14) \cap (4.17)$,
$(4.15) \cap (4.16)$, $(4.15) \cap (4.18)$, $(4.16) \cap (4.19)$, $(4.17) \cap (4.20)$, $(4.19) \cap (4.20)$

which are disjoint sets. Therefore the number of repeated redundant elements, even for $p = 2m, 2m+1$, is the sum of the cardinality of such sets. By (4.23)–(4.31) this number equals

$$\begin{aligned}
&6(p-m+1)+6(p-2m+1)(p-2m+2)+6(p-2m)(p-2m+1)\\
&+3(p-2m)(p-2m+1)+6(p-2m)(p-2m+1)+6(p-2m)(p-2m+1)\\
&+6(p-2m)(p-2m-1)+\frac{3}{2}(p-2m)(p-2m-1)+6(p-2m)\\
&=6(p-m+1)+6(p-2m+1)(p-2m+2)+21(p-2m)(p-2m+1)\\
&+\frac{15}{2}(p-2m)(p-2m-1)+6(p-2m)\\
&=6(p-m+1)+6(p-2m+1)(p-2m+2)+\frac{3}{2}(p-2m)(19p-38m+13)\\
&=6(p-m)+6+6(p-2m)^2+6(p-2m)+12(p-2m)+12\\
&+\frac{3\cdot 19}{2}(p-2m)^2+\frac{3\cdot 13}{2}(p-2m)\\
&=18+6(p-m)+\frac{75}{2}(p-2m)+\frac{69}{2}(p-2m)^2 \qquad (4.32)
\end{aligned}$$

□

**Proposition 4.3.3.** *For each $2m \le p \le 3m-1$*

$$\dim(A^p(\mathbb{C}^m[4])) = 6+\frac{25}{2}p-\frac{75}{2}m+\frac{15}{2}p^2-45pm+\frac{135}{2}m^2 \qquad (4.33)$$

*Proof.* The dimension of the $p$-cocycles when $2m \le p \le 3m-1$ is obtained subtracting the number of the conditions which appear in the cases $p = m, m+1, m+2, 2m, 2m+1$ (i.e. (4.13′)–(4.20′)) from the value of $d_p$. As observed in previous lemma, we must take care of (4.22) in finding the dimension of the Chow groups $A^p(\mathbb{C}^m[4])$, hence

$$\begin{aligned}
&\dim(A^p(\mathbb{C}^m[4]))\\
&= d_p(\mathbb{C}^m[4]))\\
&-6(p-m+1)^2-\frac{3}{2}(p-2m+2)(p+2m+1)-3m(m-1)\\
&-6(p-m)(p-m+1)-4(p-m)-6(p-m)(p-m-1)\\
&-\frac{3}{2}(p-m)(p-m-1)\\
&-4(p-2m+1)-6(p-2m+1)(p-2m)\\
&-6(p-2m)^2-6(p-2m)-\frac{3}{2}(p-2m)(p-2m-1)\\
&+18+6(p-m)+\frac{75}{2}(p-2m)+\frac{69}{2}(p-2m)^2
\end{aligned}$$

Simplifying this expression we find

$$\begin{aligned}
&\dim(A^p(\mathbb{C}^m[4])) \\
&= 1 + \frac{5}{2}p(1+3p) - \frac{39}{2}(p-m)^2 - \frac{17}{2}(p-m) + 21(p-2m)^2 \\
&\quad - \frac{1}{2}(p-2m)(3p+6m-43) - 3(p+m+m^2) + 5
\end{aligned}$$

After some more calculation, we finally arrive at the form

$$\dim(A^p(\mathbb{C}^m[4])) = 6 + \frac{25}{2}p - \frac{75}{2}m + \frac{15}{2}p^2 - 45pm + \frac{135}{2}m^2$$

□

Moreover we may state the following

**Theorem 4.3.1.** *There is a symmetry among the Chow groups with respect to the codimension $[\frac{3m-1}{2}]$, that is $\forall\ 0 \le p \le [\frac{3m-1}{2}]$*

$$\dim(A^p(\mathbb{C}^m[4])) = \dim(A^{3m-1-p}(\mathbb{C}^m[4])) \tag{4.34}$$

*In particular $\dim(A^p(\mathbb{C}^m[4]))$ does not depend on $m$, for $2m \le p \le 3m-1$.*

*Proof.* If $0 \le p \le m-1$ then $2m \le 3m-1-p \le 3m-1$. Therefore $\dim(A^p(\mathbb{C}^m[4]))$ is provided by (4.11), whereas in order to obtain $\dim(A^{3m-1-p}(\mathbb{C}^m[4]))$ we apply (4.33) and in both the cases the dimension is

$$1 + \frac{15}{2}p^2 + \frac{5}{2}p$$

If $m \le p \le [\frac{3m-1}{2}]$ then $[\frac{3m-1}{2}] \le 3m-1-p \le 2m-1$ hence both $\dim(A^p(\mathbb{C}^m[4]))$ and $\dim(A^{3m-1-p}(\mathbb{C}^m[4]))$ are given by (4.21). Expanding (4.21) in $p$ and in $3m-1-p$ provides the same expression

$$\frac{35}{2}m - \frac{45}{2}m^2 + 45pm - 15p - 15p^2 - 5$$

□

**Proposition 4.3.4.** *For all $p \ge 3m$ the $p$-cocycles of $\mathbb{C}^m[4]$ are rationally equivalent to zero*

$$A^p(\mathbb{C}^m[4]) = 0 \tag{4.35}$$

*Proof.* From Proposition 4.3.3 we deduce that $\dim(A^{3m-1}(\mathbb{C}^m[4])) = 1$ and it is obvious that $A^{3m-1}(\mathbb{C}^m[4]) = \mathbb{Z}[D_{1234}^{3m-1}]$.

To obtain the $3m$-cocycles we intersect $D_{1234}^{3m-1}$ with all the divisors $D_S$.

Since $D_{ijh} \cdot D_{1234}^m = 0$, $D_{ij} \cdot D_{1234}^{2m} = 0$ and $D_{1234}^{3m} = 0$, the group vanishes and the same holds for $p > 3m$. □

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